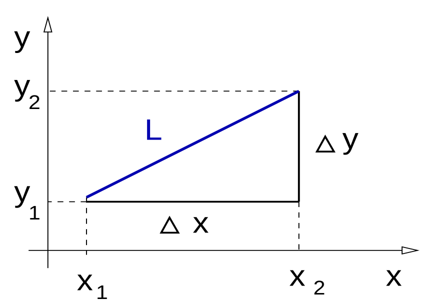
The arc-length of curves in the plane (Sect. 6.3)

- The main arc-length formula.
- Curves with vertical asymptotes.
- The arc-length function.

The main length formula

Remark: The length of a straight segment can be obtained with Pythagoras Theorem.

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$



Remark: Calculus is needed to compute, and even define, the length of non-straight curves, called arc-length.

Definition

The *arc-length* of a curve in the plane given by a differentiable function $y = f(x)$, for $x \in [a, b]$, is

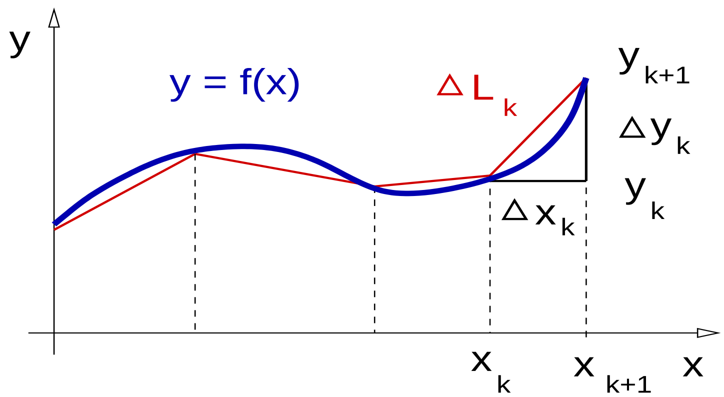
$$L = \int_a^b \sqrt{1 + \big[f'(x)\big]^2}\, dx.$$

The main length formula

Remark: The origin of the square-root in the expression above is Pythagoras Theorem.

Remark: The definition of arc-length is the result of a limit procedure. We mention two of such limits.

(a)



The length of the curve will be approximated by the red lines,

$$L_N = \sum_{k=0}^{N-1} \Delta L_k = \sum_{k=0}^{N-1} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

The main length formula

$$L_N = \sum_{k=0}^{N-1} \Delta L_k = \sum_{k=0}^{N-1} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

$$L_N = \sum_{k=0}^{N-1} \sqrt{1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}}\, \Delta x_k = \sum_{k=0}^{N-1} \sqrt{1 + \frac{(y_{k+1} - y_k)^2}{(x_{k+1} - x_k)^2}}\, \Delta x_k$$

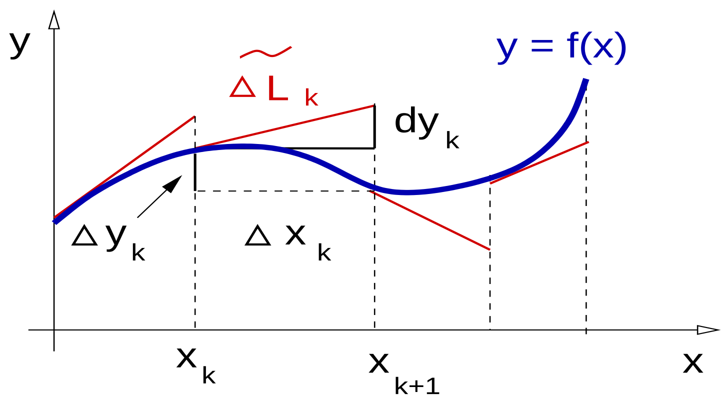
$$L_N = \sum_{k=0}^{N-1} \sqrt{1 + \Big[\frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}\Big]^2}\, \Delta x_k$$

One can show that in the limit $N \to \infty$ holds $x_{k+1} \to x_k$ and

$$L_N \to \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = L.$$

The main length formula

(b)



Recall: $dy_k = f'(x_k)\,\Delta x_k$. Now the length of the curve will be approximated by the red lines

$$\tilde{L}_N = \sum_{k=0}^{N-1} \widetilde{\Delta L_k} = \sum_{k=0}^{N-1} \sqrt{(\Delta x_k)^2 + (dy_k)^2}.$$

The main length formula

$$\tilde{L}_N = \sum_{k=0}^{N-1} \widetilde{\Delta L_k} = \sum_{k=0}^{N-1} \sqrt{(\Delta x_k)^2 + (dy_k)^2}.$$

$$\tilde{L}_N = \sum_{k=0}^{N-1} \sqrt{1 + \frac{(dy_k)^2}{(\Delta x_k)^2}}\,\Delta x_k = \sum_{k=0}^{N-1} \sqrt{1 + \frac{\left[f'(x_k)\Delta x_k\right]^2}{(\Delta x_k)^2}}\,\Delta x_k$$

$$\tilde{L}_N = \sum_{k=0}^{N-1} \sqrt{1 + \left[f'(x_k)\right]^2}\,\Delta x_k$$

One can show that in the limit $N \to \infty$ holds

$$\tilde{L}_N \to \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx = L.$$

The main length formula

Example

Find the arc-length of the curve $y = x^{3/2}$, for $x \in [0, 4]$.

Solution: Recall: $L = \int_a^b \sqrt{1 + \big[f'(x)\big]^2}\, dx$. We start with

$$f(x) = x^{3/2} \quad \Rightarrow \quad f'(x) = \frac{3}{2}\, x^{1/2} \quad \Rightarrow \quad \big[f'(x)\big]^2 = \frac{9}{4}\, x.$$

$$L = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx, \quad u = 1 + \frac{9}{4}\, x, \quad du = \frac{9}{4}\, dx.$$

$$L = \int_1^{10} \frac{4}{9}\, \sqrt{u}\, du = \frac{4}{9}\, \frac{2}{3}\, \Big(u^{3/2}\Big|_1^{10}\Big).$$

We conclude that $L = \dfrac{8}{27}(10^{3/2} - 1)$. $\triangleleft$

The arc-length of curves in the plane (Sect. 6.3)

- The main arc-length formula.
- **Curves with vertical asymptotes.**
- The arc-length function.

Curves with vertical asymptotes

Remark: The arc-length of curves having a vertical asymptote should be computed using the inverse function.

Example

Find the arc-length of $y(x) = \sqrt{2(x-1)}$, for $x \in [1,3]$.

Solution: Recall: $L = \int_a^b \sqrt{1 + \big[y'(x)\big]^2}\, dx$.

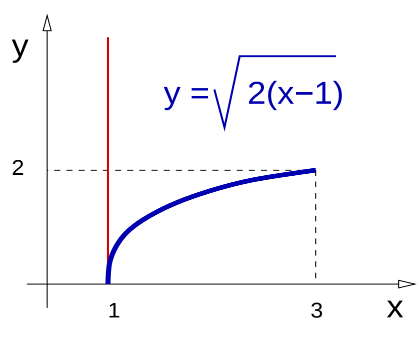
$$y'(x) = \sqrt{2}\big(\sqrt{x-1}\big)' = \sqrt{2}\,\frac{1}{2}\,\frac{1}{\sqrt{x-1}} = \frac{1}{\sqrt{2(x-1)}}.$$

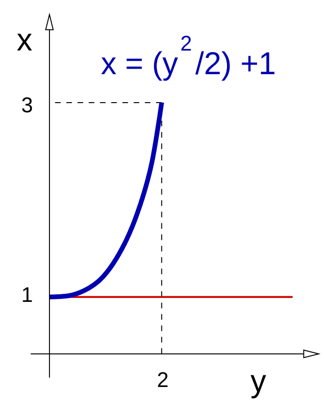
Hence, $y'(x) \to \infty$ as $x \to 1^+$. Therefore, it is not clear how to compute

$$L = \int_1^3 \sqrt{1 + \frac{1}{2(x-1)}}\, dx.$$

Curves with vertical asymptotes

Remark: Describe the curve with the inverse function.





We now use $L = \int_0^2 \sqrt{1 + \big[x'(y)\big]^2}\, dy$. Since $x'(y) = y$,

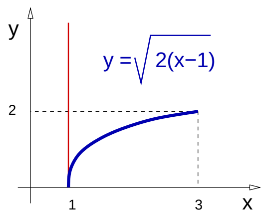
$$L = \int_0^2 \sqrt{1+y^2}\, dy = \Big[\frac{y}{2}\sqrt{1+y^2} + \frac{1}{2}\ln\big(y + \sqrt{1+y^2}\big)\Big]\Big|_0^2.$$

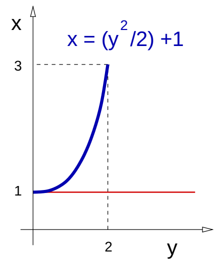
Curves with vertical asymptotes

Example

Find the length of $y(x) = \sqrt{2(x-1)}$, for $x \in [1,3]$.

Solution: Recall:





$$L = \int_0^2 \sqrt{1+y^2}\,dy = \Big[\frac{y}{2}\sqrt{1+y^2} + \frac{1}{2}\ln\big(y + \sqrt{1+y^2}\big)\Big]\Big|_0^2.$$

We conclude that $L = \sqrt{5} + \frac{1}{2}\ln\big(2+\sqrt{5}\big)$. $\triangleleft$

The arc-length of curves in the plane (Sect. 6.3)

- The main arc-length formula.
- Curves with vertical asymptotes.
- **The arc-length function.**

The arc-length function.

Remark: It is useful to introduce a function that measures a curve arc-length from a fix starting point to any other point in the curve.

Definition

The *arc-length function* of a differentiable curve $y = f(x)$, for $x \in [a, b]$ is given by

$$L(x) = \int_a^x \sqrt{1 + \left[f'(\hat{x})\right]^2}\, d\hat{x}.$$

Remark: The Fundamental Theorem of Calculus implies that

$$L'(x) = \sqrt{1 + \left[f'(x)\right]^2}$$

Remark: Using differential notation, $dL = L'(x)\, dx$, we get

$$dL = \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

The arc-length function.

Example

Find the arc-length function of the curve $y = x^{3/2}$, for $x \in [0, 4]$.

Solution: Recall: $f'(x) = \frac{3}{2}\, x^{1/2}$, so $\left[f'(x)\right]^2 = \frac{9}{4}\, x$.

$$L(x) = \int_0^x \sqrt{1 + \frac{9}{4}\tilde{x}}\, d\tilde{x}, \quad u = 1 + \frac{9}{4}\,\tilde{x}, \quad du = \frac{9}{4}\, d\tilde{x}.$$

$$L(x) = \int_1^{1+\frac{9}{4}x} \frac{4}{9}\sqrt{u}\, du = \frac{4}{9}\,\frac{2}{3}\left(u^{3/2}\Big|_1^{1+\frac{9}{4}x}\right).$$

We conclude that $L(x) = \dfrac{8}{27}\left[\left(1 + \dfrac{9}{4}\, x\right)^{3/2} - 1\right].$ $\triangleleft$