Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
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Recall:

The point $x_0 \in \mathbb{R}$ is a singular point of the equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

iff holds that $P(x_0) = 0$.

Equations with regular-singular points.

Definition

A singular point $x_0 \in \mathbb{R}$ of the equation

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

is called a *regular-singular* point iff the following limits are finite,

$$\lim_{x \to x_0} \frac{(x - x_0)\, Q(x)}{P(x)}, \qquad \lim_{x \to x_0} \frac{(x - x_0)^2\, R(x)}{P(x)},$$

and both functions

$$\frac{(x - x_0)\, Q(x)}{P(x)}, \qquad \frac{(x - x_0)^2\, R(x)}{P(x)},$$

admit convergent Taylor series expansions around x_0.

Equations with regular-singular points.

Remark:

- If x_0 is a regular-singular point of

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

and $P(x) \simeq (x - x_0)^n$ near x_0, then near x_0 holds

$$Q(x) \simeq (x - x_0)^{n-1}, \quad R(x) \simeq (x - x_0)^{n-2}.$$

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- The main example is an Euler equation, case $n = 2$,

$$(x - x_0)^2\,y'' + p_0(x - x_0)\,y' + q_0\,y = 0.$$

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Show that the singular point of every Euler equation is a regular-singular point.

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Solution: Consider the general Euler equation

$$(x - x_0)^2\, y'' + p_0(x - x_0)\, y' + q_0\, y = 0,$$

where p_0, q_0, x_0, are real constants.

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We conclude that x_0 is a regular-singular point. $\triangleleft$

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Remark: Every equation $Py'' + Qy' + Ry = 0$ with a regular-singular point at x_0 is close to an Euler equation.

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Furthermore, the following limits are finite,

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Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$(1 - x^2)\, y'' - 2x\, y' + \alpha(\alpha + 1)\, y = 0,$$

where α is a real constant.

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Both functions have Taylor series around $x_1 = 1$.

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and the log function does not have a Taylor series at $x_0 = 0$.

Examples: Equations with regular-singular points.

Example

Find the regular-singular points of the differential equation

$$x\,y'' - x\ln(|x|)\,y' + 3x\,y = 0.$$

Solution: Recall: $\displaystyle\lim_{x\to 0} \frac{xQ(x)}{P(x)} = 0$ and $\displaystyle\lim_{x\to 0} \frac{x^2 R(x)}{P(x)} = 0$.

However, at the point $x_0 = 0$ the function xQ/P does not have a power series expansion around zero, since

$$\frac{xQ(x)}{P(x)} = -x\ln(|x|),$$

and the log function does not have a Taylor series at $x_0 = 0$.

We conclude that $x_0 = 0$ is not a regular-singular point. ◁

Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- **Method to find solutions.**
- Example: Method to find solutions.

Method to find solutions.

Recall: If x_0 is a regular-singular point of

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0,$$

with limits $\displaystyle\lim_{x\to x_0} \frac{(x-x_0)Q(x)}{P(x)} = p_0$ and $\displaystyle\lim_{x\to x_0} \frac{(x-x_0)^2 R(x)}{P(x)} = q_0$,

then the coefficients of the differential equation above near x_0 are close to the coefficients of the Euler equation

$$(x-x_0)^2\,y'' + p_0(x-x_0)\,y' + q_0\,y = 0.$$

Method to find solutions.

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Idea: If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

Method to find solutions.

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$$(x-x_0)^2\, y'' + p_0(x-x_0)\, y' + q_0\, y = 0.$$

Idea: If the differential equation is close to an Euler equation, then the solutions of the differential equation might be close to the solutions of an Euler equation.

Recall: One solution of an Euler equation is $y(x) = (x-x_0)^r$.

Method to find solutions.

Summary: Solutions for equations with regular-singular points:

Method to find solutions.

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(1) Look for a solution y of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^{(n+r)};$$

Method to find solutions.

Summary: Solutions for equations with regular-singular points:

(1) Look for a solution y of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^{(n+r)};$$

(2) Introduce this power series expansion into the differential equation and find both a the exponent r and a recurrence relation for the coefficients a_n;

Method to find solutions.

Summary: Solutions for equations with regular-singular points:

(1) Look for a solution y of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^{(n+r)};$$

(2) Introduce this power series expansion into the differential equation and find both a the exponent r and a recurrence relation for the coefficients a_n;

(3) First find the solutions for the constant r. Then, introduce this result for r into the recurrence relation for the coefficients a_n. Only then, solve this latter recurrence relation for the coefficients a_n.

Equations with regular-singular points (Sect. 5.5).

- Equations with regular-singular points.
- Examples: Equations with regular-singular points.
- Method to find solutions.
- **Example: Method to find solutions.**

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n\, x^{(n+r)}$.

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We look for a solution $y(x) = \sum_{n=0}^{\infty} a_n\, x^{(n+r)}$.

The first and second derivatives are given by

$$y' = \sum_{n=0}^{\infty} (n+r) a_n\, x^{(n+r-1)},$$

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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The first and second derivatives are given by

$$y' = \sum_{n=0}^{\infty} (n+r) a_n\, x^{(n+r-1)}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r-2)}.$$

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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In the case $r = 0$ we had the relation

$$\sum_{n=0}^{\infty} n a_n\, x^{(n-1)} = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)},$$

Example: Method to find solutions.

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In the case $r = 0$ we had the relation

$$\sum_{n=0}^{\infty} n a_n\, x^{(n-1)} = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)},$$

but for $r \neq 0$ this relation is not true.

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We now compute the term $(x+3)y$,

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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$$(x+3)\, y = (x+3) \sum_{n=0}^{\infty} a_n\, x^{(n+r)}$$

Example: Method to find solutions.

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$$(x+3)\, y = (x+3) \sum_{n=0}^{\infty} a_n\, x^{(n+r)}$$

$$(x+3)\, y = \sum_{n=0}^{\infty} a_n\, x^{(n+r+1)} + \sum_{n=0}^{\infty} 3a_n\, x^{(n+r)}$$

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Example

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$$(x+3)\, y = \sum_{n=0}^{\infty} a_n\, x^{(n+r+1)} + \sum_{n=0}^{\infty} 3a_n\, x^{(n+r)}$$

$$(x+3)\, y = \sum_{n=1}^{\infty} a_{(n-1)}\, x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n\, x^{(n+r)}.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3) y' + (x+3) y = 0.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We now compute the term $-x(x+3)\, y'$,

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We now compute the term $-x(x+3)\, y'$,

$$-x(x+3)\, y' = -(x^2 + 3x) \sum_{n=0}^{\infty} (n+r) a_n\, x^{(n+r-1)}$$

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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$$-x(x+3)\,y' = -(x^2+3x)\sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r-1)}$$

$$-x(x+3)\,y' = -\sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r+1)} - \sum_{n=0}^{\infty}3(n+r)a_n\,x^{(n+r)},$$

Example: Method to find solutions.

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$$-x(x+3)\,y' = -\sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\,x^{(n+r)} - \sum_{n=0}^{\infty}3(n+r)a_n\,x^{(n+r)}.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\,y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

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Example

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$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We compute the term $x^2\, y''$,

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\,y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

Solution: We compute the term $x^2\,y''$,

$$x^2\,y'' = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n\,x^{(n+r-2)}$$

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We compute the term $x^2\, y''$,

$$x^2\, y'' = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r-2)}$$

$$x^2\, y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r)}.$$

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Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We compute the term $x^2\, y''$,

$$x^2\, y'' = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r-2)}$$

$$x^2\, y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n\, x^{(n+r)}.$$

The guiding principle to rewrite each term is to have the power function $x^{(n+r)}$ labeled in the same way on every term.

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: The differential equation is given by

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: The differential equation is given by

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\, x^{(n+r)} - \sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\, x^{(n+r)}$$

$$-\sum_{n=0}^{\infty}3(n+r)a_n\, x^{(n+r)} + \sum_{n=1}^{\infty}a_{(n-1)}\, x^{(n+r)} + \sum_{n=0}^{\infty}3a_n\, x^{(n+r)} = 0.$$

Example: Method to find solutions.

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We split the sums into the term $n = 0$ and a sum containing the terms with $n \geqslant 1$,

Example: Method to find solutions.

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$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\, x^{(n+r)} - \sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\, x^{(n+r)}$$

$$- \sum_{n=0}^{\infty} 3(n+r)a_n\, x^{(n+r)} + \sum_{n=1}^{\infty} a_{(n-1)}\, x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n\, x^{(n+r)} = 0.$$

We split the sums into the term $n = 0$ and a sum containing the terms with $n \geqslant 1$, that is,

$$0 = \big[r(r-1) - 3r + 3\big]\, a_0 x^r +$$

$$\sum_{n=1}^{\infty}\Big[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n\Big]\, x^{(n+r)}$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Therefore, $\big[r(r-1) - 3r + 3\big] = 0$ and

$$\Big[(n{+}r)(n{+}r{-}1)a_n - (n{+}r{-}1)a_{(n-1)} - 3(n{+}r)a_n + a_{(n-1)} + 3a_n\Big] = 0.$$

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Therefore, $\big[r(r-1) - 3r + 3\big] = 0$ and

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The last expression can be rewritten as follows,

$$\Big[\big[(n+r)(n+r-1) - 3(n+r) + 3\big]\, a_n - (n+r-1-1)a_{(n-1)}\Big] = 0,$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

Solution: Therefore, $\big[r(r-1) - 3r + 3\big] = 0$ and

$$\Big[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n\Big] = 0.$$

The last expression can be rewritten as follows,

$$\Big[\big[(n+r)(n+r-1) - 3(n+r) + 3\big]\,a_n - (n+r-1-1)a_{(n-1)}\Big] = 0,$$

$$\Big[\big[(n+r)(n+r-1) - 3(n+r-1)\big]\,a_n - (n+r-2)a_{(n-1)}\Big] = 0.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Hence, the recurrence relation is given by the equations

$$r(r-1) - 3r + 3 = 0,$$
$$(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)} = 0.$$

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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First: solve the first equation for $r_{\pm}$.

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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First: solve the first equation for $r_\pm$.

Second: Introduce the first solution r_+ into the second equation above and solve for the a_n;

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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First: solve the first equation for $r_\pm$.

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Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

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$$r(r-1) - 3r + 3 = 0,$$
$$(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)} = 0.$$

First: solve the first equation for $r_\pm$.

Second: Introduce the first solution r_+ into the second equation above and solve for the a_n; the result is a solution y_+ of the original differential equation;

Third: Introduce the second solution r_- into into the second equation above and solve for the a_n;

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Hence, the recurrence relation is given by the equations

$$r(r-1) - 3r + 3 = 0,$$
$$(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)} = 0.$$

First: solve the first equation for $r_\pm$.

Second: Introduce the first solution r_+ into the second equation above and solve for the a_n; the result is a solution y_+ of the original differential equation;

Third: Introduce the second solution r_- into into the second equation above and solve for the a_n; the result is a solution y_- of the original differential equation;

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We first solve $r(r-1) - 3r + 3 = 0$.

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We first solve $r(r-1) - 3r + 3 = 0$.

$r^2 - 4r + 3 = 0$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We first solve $r(r-1) - 3r + 3 = 0$.

$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-12}\big]$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We first solve $r(r-1) - 3r + 3 = 0$.

$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-12}\big] \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = 1. \end{cases}$$

Example: Method to find solutions.

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Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: We first solve $r(r-1) - 3r + 3 = 0$.

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Introduce $r_+ = 3$ into the equation for a_n:

$$(n+2)n\, a_n - (n+1)a_{n-1} = 0.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\,y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

Solution: We first solve $r(r-1) - 3r + 3 = 0$.

$$r^2 - 4r + 3 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-12}\big] \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = 1. \end{cases}$$

Introduce $r_+ = 3$ into the equation for a_n:

$$(n+2)n\,a_n - (n+1)a_{n-1} = 0.$$

One can check that the solution y_+ is

$$y_+ = a_0\,x^3\Big[1 + \frac{2}{3}\,x + \frac{1}{4}\,x^2 + \frac{1}{15}\,x^3 + \cdots\Big].$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Introduce $r_- = 1$ into the equation for a_n:

$$n(n-2)a_n - (n-1)a_{n-1} = 0.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Introduce $r_- = 1$ into the equation for a_n:

$$n(n-2)a_n - (n-1)a_{n-1} = 0.$$

One can also check that the solution y_- is

$$y_- = a_2\, x\Big[x^2 + \frac{2}{3}\, x^3 + \frac{1}{4}\, x^4 + \frac{1}{15}\, x^5 + \cdots\Big].$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Introduce $r_- = 1$ into the equation for a_n:

$$n(n-2)a_n - (n-1)a_{n-1} = 0.$$

One can also check that the solution y_- is

$$y_- = a_2\, x\Big[x^2 + \frac{2}{3}\, x^3 + \frac{1}{4}\, x^4 + \frac{1}{15}\, x^5 + \cdots\Big].$$

Notice:

$$y_- = a_2\, x^3\Big[1 + \frac{2}{3}\, x + \frac{1}{4}\, x^2 + \frac{1}{15}\, x^3 + \cdots\Big]$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2\, y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: Introduce $r_- = 1$ into the equation for a_n:

$$n(n-2)a_n - (n-1)a_{n-1} = 0.$$

One can also check that the solution y_- is

$$y_- = a_2\, x \Big[x^2 + \frac{2}{3}\, x^3 + \frac{1}{4}\, x^4 + \frac{1}{15}\, x^5 + \cdots \Big].$$

Notice:

$$y_- = a_2\, x^3 \Big[1 + \frac{2}{3}\, x + \frac{1}{4}\, x^2 + \frac{1}{15}\, x^3 + \cdots \Big] \;\Rightarrow\; y_- = \frac{a_2}{a_1}\, y_+.$$

Example: Method to find solutions.

Example

Find the solution y near the regular-singular point $x_0 = 0$ of

$$x^2 y'' - x(x+3)\, y' + (x+3)\, y = 0.$$

Solution: The solutions y_+ and y_- are not linearly independent.

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Remark: It can be shown the following result:
If the roots of the Euler characteristic polynomial r_+, r_- differ by an integer, then the second solution y_-, the solution corresponding to the smaller root, is not given by the method above.

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This solution involves logarithmic terms.
We do not study this type of solutions in these notes. $\triangleleft$

The Laplace Transform (Sect. 6.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
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The definition of the Laplace Transform.

Definition

The function $F : D_F \to \mathbb{R}$ is the *Laplace transform* of a function $f : [0, \infty) \to \mathbb{R}$ iff for all $s \in D_F$ holds,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

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- Functions are denoted as $t \mapsto f(t)$.
- The Laplace transform is also a function: $f \mapsto \mathcal{L}[f]$.

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Recall: Improper integral are defined as a limit.

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Compute $\mathcal{L}[1]$.

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Therefore $\mathcal{L}[1] = \dfrac{1}{s}$, for $s > 0$, and $\mathcal{L}[1]$ does not exists for $s \leqslant 0$.

In other words, $F(s) = \mathcal{L}[1]$ is the function $F : D_F \to \mathbb{R}$ given by

$$f(t) = 1, \qquad F(s) = \frac{1}{s}, \qquad D_F = (0, \infty). \qquad \triangleleft$$

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$$f(t) = e^{at}, \qquad F(s) = \frac{1}{(s-a)}, \qquad s > a. \qquad \triangleleft$$

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This identity implies

$$\left(1+\frac{a^2}{s^2}\right)\int_0^N e^{-st}\sin(at)\, dt = -\frac{1}{s}\left[e^{-st}\sin(at)\right]\Big|_0^N - \frac{a}{s^2}\left[e^{-st}\cos(at)\right]\Big|_0^N.$$

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Hence, it is not difficult to see that

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which is equivalent to

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A table of Laplace Transforms.

$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0,$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{s-a}$	$s > \max\{a, 0\},$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0,$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0,$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0,$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > 0,$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > 0,$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > \max\{a, 0\},$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > \max\{a, 0\}.$

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Theorem (Sufficient conditions)

If the function $f : [0, \infty) \to \mathbb{R}$ *is piecewise continuous and there exist positive constants* k *and* a *such that*

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Theorem (Linear combination)

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If the $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are well-defined, then holds,

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$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0). \tag{2}$$

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Proof of Eq (2): Use Eq. (1) twice:

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Properties of the Laplace Transform.

Theorem (Derivatives)

If the $\mathcal{L}[f]$ and $\mathcal{L}[f']$ are well-defined, then holds,

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$$\lim_{n\to\infty} \int_0^n e^{-st} f'(t)\, dt = \lim_{n\to\infty} \Big[\Big(e^{-st} f(t) \Big)\Big|_0^n - \int_0^n (-s) e^{-st} f(t)\, dt \Big]$$

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where we used that $\lim_{n\to\infty} e^{-sn} f(n) = 0$ for s big enough, and we also used that $\mathcal{L}[f]$ is well-defined.

We then conclude that $\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)$.

The Laplace Transform (Sect. 6.1).

- The definition of the Laplace Transform.
- Review: Improper integrals.
- Examples of Laplace Transforms.
- A table of Laplace Transforms.
- Properties of the Laplace Transform.
- **Laplace Transform and differential equations.**

Laplace Transform and differential equations.

Remark: Laplace Transforms can be used to find solutions to differential equations with constant coefficients.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y' + 2y = 0, \qquad y(0) = 3.$$

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Find an algebraic equation for $\mathcal{L}[y]$.

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$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}]$

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Hence, $\mathcal{L}[y] = \mathcal{L}[3e^{-2t}] \quad \Rightarrow \quad y(t) = 3e^{-2t}$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - Non-homogeneous IVP.
  - Recall: Partial fraction decompositions.

Solving differential equations using $\mathcal{L}[\ ]$.

Remark: The method works with:

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Recall:

(a) $\mathcal{L}[a\,f(t) + b\,g(t)] = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$;

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(b) $\mathcal{L}\big[y^{(n)}\big] = s^n\, \mathcal{L}[y] - s^{(n-1)}\, y(0) - s^{(n-2)}\, y'(0) - \cdots - y^{(n-1)}(0)$.

The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - **Homogeneous IVP.**
  - First, second, higher order equations.
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  - Recall: Partial fraction decompositions.

Homogeneous IVP.

Example

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$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

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Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

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The $\mathcal{L}[\ ]$ is a linear function,

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The $\mathcal{L}[\;]$ is a linear function, so

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0.$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

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We the obtain $(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0)$.

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Differential equation for y $\xrightarrow{\mathcal{L}[\ ]}$ Algebraic equation for $\mathcal{L}[y]$.

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Homogeneous IVP.

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We can solve for the unknown $\mathcal{L}[y]$ as follows,

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Introduce the initial condition,

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We can solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{(s-1)}{(s^2 - s - 2)}.$$

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The partial fraction method: Find the zeros of the denominator,

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$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[1 \pm \sqrt{1+8}\right]$$

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Therefore, we rewrite: $\mathcal{L}[y] = \dfrac{(s-1)}{(s-2)(s+1)}$.

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Therefore, we rewrite: $\mathcal{L}[y] = \dfrac{(s-1)}{(s-2)(s+1)}$.

Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

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$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$$

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$$(s-1) = s(a+b) + (a-2b)$$

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Hence, $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$.

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$$(s-1) = s(a+b) + (a-2b) \quad \Rightarrow \quad \begin{cases} a + b = 1, \\ a - 2b = -1 \end{cases}$$

Hence, $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$. Then, $\mathcal{L}[y] = \dfrac{1}{3}\,\dfrac{1}{(s-2)} + \dfrac{2}{3}\,\dfrac{1}{(s+1)}$.

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Solution: Recall: $\mathcal{L}[y] = \frac{1}{3}\,\frac{1}{(s-2)} + \frac{2}{3}\,\frac{1}{(s+1)}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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So we arrive at the equation

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We conclude that: $y(t) = \frac{1}{3}\big(e^{2t} + 2e^{-t}\big)$. $\triangleleft$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

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$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

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The $\mathcal{L}[\ ]$ is a linear function,

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Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = 0,$$

Therefore, $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0).$

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Introduce the initial conditions,

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Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0)$.

Introduce the initial conditions, $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3$.

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Solve for $\mathcal{L}[y]$ as follows:

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Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)}$.

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$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\left[4 \pm \sqrt{16 - 16}\right]$$

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$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\big[4 \pm \sqrt{16 - 16}\big] \quad \Rightarrow \quad s_+ = s_- = 2.$$

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$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\left[4 \pm \sqrt{16 - 16}\right] \quad \Rightarrow \quad s_+ = s_- = 2.$$

We obtain: $\mathcal{L}[y] = \dfrac{(s-3)}{(s-2)^2}$.

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This expression is already in the partial fraction decomposition.

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This expression is already in the partial fraction decomposition.
Idea: Rewrite the right-hand side in terms of function in the table.

$$\mathcal{L}[y] = \frac{(s-2) + 2 - 3}{(s-2)^2}$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-3)}{(s-2)^2}$.

This expression is already in the partial fraction decomposition.
Idea: Rewrite the right-hand side in terms of function in the table.

$$\mathcal{L}[y] = \frac{(s-2)+2-3}{(s-2)^2} = \frac{(s-2)}{(s-2)^2} - \frac{1}{(s-2)^2}$$

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From the Laplace transforms table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

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$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{(n+1)}}$$

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Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{1}{s-2} - \dfrac{1}{(s-2)^2}$ and

$$\frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}]$$

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So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}] = \mathcal{L}\big[e^{2t} - te^{2t}\big].$$

We conclude that $y(t) = e^{2t} - te^{2t}$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - **First, second, higher order equations.**
  - Non-homogeneous IVP.
  - Recall: Partial fraction decompositions.

First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

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$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

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$$\mathcal{L}\big[y^{(4)}\big] - 4\,\mathcal{L}[y] = 0.$$

First, second, higher order equations.

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$$\big[s^4\,\mathcal{L}[y] - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0)\big] - 4\,\mathcal{L}[y] = 0.$$

First, second, higher order equations.

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$$\big[s^4\,\mathcal{L}[y] - s^3 + 2s\big] - 4\,\mathcal{L}[y] = 0$$

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$$\big[s^4\,\mathcal{L}[y] - s^3 + 2s\big] - 4\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\,\mathcal{L}[y] = s^3 - 2s,$$

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$$\left[s^4\,\mathcal{L}[y] - s^3 + 2s\right] - 4\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\,\mathcal{L}[y] = s^3 - 2s,$$

We obtain, $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

First, second, higher order equations.

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Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

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$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)}$$

First, second, higher order equations.

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Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

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The last expression is in the table of Laplace Transforms,

First, second, higher order equations.

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The last expression is in the table of Laplace Transforms,

$$\mathcal{L}[y] = \frac{s}{\left(s^2 + \left[\sqrt{2}\right]^2\right)}$$

First, second, higher order equations.

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Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

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The last expression is in the table of Laplace Transforms,

$$\mathcal{L}[y] = \frac{s}{\left(s^2 + \left[\sqrt{2}\right]^2\right)} = \mathcal{L}\left[\cos(\sqrt{2}\,t)\right].$$

First, second, higher order equations.

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We conclude that $y(t) = \cos(\sqrt{2}\,t)$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\ ]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - **Non-homogeneous IVP.**
  - Recall: Partial fraction decompositions.

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the Laplace transform of the equation,

Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the Laplace transform of the equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\sin(2t)].$$

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The right-hand side above can be expressed as follows,

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The right-hand side above can be expressed as follows,

$$\mathcal{L}[3\sin(2t)] = 3\,\mathcal{L}[\sin(2t)] = 3\,\frac{2}{s^2 + 2^2}$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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The right-hand side above can be expressed as follows,

$$\mathcal{L}[3\sin(2t)] = 3\,\mathcal{L}[\sin(2t)] = 3\,\frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

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Introduce this source term in the differential equation,

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

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Introduce this source term in the differential equation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \dfrac{6}{s^2 + 4}$.

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \dfrac{6}{s^2+4}$.

Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{6}{s^2+4}.$$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Rewrite the above equation,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0) + \frac{6}{s^2+4}.$$

Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

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Rewrite the above equation,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0) + \frac{6}{s^2+4}.$$

Introduce the initial conditions,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \frac{6}{s^2+4}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \dfrac{6}{s^2+4}.$

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \dfrac{6}{s^2 + 4}$.

Therefore, $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)} + \dfrac{6}{(s^2 - 4 + 4)(s^2 + 4)}$.

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \dfrac{6}{s^2 + 4}$.

Therefore, $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)} + \dfrac{6}{(s^2 - 4 + 4)(s^2 + 4)}$.

From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

Non-homogeneous IVP.

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Therefore, $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2-4s+4)} + \dfrac{6}{(s^2-4+4)(s^2+4)}$.

From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2} + \frac{6}{(s-2)^2(s^2+4)}.$$

Non-homogeneous IVP.

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Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \dfrac{6}{s^2+4}$.

Therefore, $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)} + \dfrac{6}{(s^2 - 4 + 4)(s^2 + 4)}$.

From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2} + \frac{6}{(s-2)^2(s^2+4)}.$$

From an Example above we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \dfrac{6}{(s-2)^2(s^2+4)}.$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \dfrac{6}{(s-2)^2(s^2+4)}$.

Use Partial fractions to simplify the last term above.

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We obtain the system

$$a + c = 0, \qquad -4a + b - 2c + d = 0,$$

$$4a - 4b + 4c = 0, \qquad 4b - 8c + 4d = 6.$$

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$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: The solution for this linear system is

$$a = \frac{3}{8}, \qquad b = 0, \qquad c = -\frac{3}{8}, \qquad d = \frac{3}{4}.$$

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We conclude that

$$y(t) = (1-t)\,e^{2t} + \frac{3}{8}\,(2t-1)\,e^{2t} + \frac{3}{8}\,\cos(2t). \qquad \triangleleft$$