# Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- Superposition property.
- Constant coefficients equations.
- The characteristic equation.
- The main result.

# Second order linear differential equations.

## Definition

Given functions $a_1$, $a_0$, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \tag{1}$$

is called a *second order linear* differential equation with *variable coefficients*. The equation in (1) is called *homogeneous* iff for all $t \in \mathbb{R}$ holds

$$b(t) = 0.$$

The equation in (1) is called of *constant coefficients* iff $a_1$, $a_0$, and $b$ are constants.

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Remark: The notion of an homogeneous equation presented here is not the same as the notion presented in the previous chapter.

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(d) Newton's second law of motion ($ma = f$) for point particles of mass $m$ moving in one space dimension under a force $f : \mathbb{R} \to \mathbb{R}$ is given by

$$m\, y''(t) = f(t).$$

# Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
- **Superposition property.**
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# Superposition property.

## Theorem

*If the functions $y_1$ and $y_2$ are solutions to the homogeneous linear equation*

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{2}$$

*then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants $c_1$, $c_2 \in \mathbb{R}$.*

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Proof: Verify that the function $y = c_1 y_1 + c_2 y_2$ satisfies Eq. (2) for every constants $c_1$, $c_2$, that is,

$$(c_1 y_1 + c_2 y_2)'' + a_1(t)(c_1 y_1 + c_2 y_2)' + a_0(t)(c_1 y_1 + c_2 y_2)$$

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$$= c_1 \big[ y_1'' + a_1(t) y_1' + a_0(t) y_1 \big] + c_2 \big[ y_2'' + a_1(t) y_2' + a_0(t) y_2 \big] = 0.$$

$\square$

# Second order linear ODE (Sect. 3.1).

- Second order linear differential equations.
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- **Constant coefficients equations.**
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This polynomial is called the characteristic polynomial of the differential equation.

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$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

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$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

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Summary: The differential equation $y'' + 5y' + 6y = 0$ has infinitely many solutions,

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- There are two free constants in the solution found above.
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- An IVP for a second order differential equation will have a unique solution if the IVP contains two initial conditions.

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- Second order linear differential equations.
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- **The characteristic equation.**
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# The characteristic equation.

## Definition

Given a second order linear homogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_0 = 0, \qquad (3)$$

the *characteristic polynomial* and the *characteristic equation* associated with the differential equation in (3) are, respectively,

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

If $r_1$, $r_2$ are the solutions of the characteristic equation and $c_1$, $c_2$ are constants, then the function

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is called the *general solution* of the Eq. (3).

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$$c_1 = 1 - c_2 \Rightarrow 1 = 2(1 - c_2) + 3c_2 \Rightarrow c_2 = -1 \Rightarrow c_1 = 2.$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

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Therefore, the general solution of the equation above is

$$y(t) = c_1 e^{t} + c_2 e^{t/2},$$

where $c_1$, $c_2$ are arbitrary constants. $\triangleleft$

# Second order linear ODE (Sect. 3.1).

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# The main result.

## Theorem (Constant coefficients)

*Given real constants $a_1$, $a_0$, consider the homogeneous, linear differential equation on the unknown $y : \mathbb{R} \to \mathbb{R}$ given by*

$$y'' + a_1\, y' + a_0\, y = 0. \tag{4}$$

*Let $r_+$, $r_-$ be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, and let $c_0$, $c_1$ be arbitrary constants. Then, any solution of Eq. (4) belongs to only one of the following cases:*

(a) *If $r_+ \neq r_-$, the general solution is $y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}$.*

(b) *If $r_+ = r_- \in \mathbb{R}$, the general solution is $y(t) = (c_0 + c_1 t)e^{r_+ t}$.*

*Furthermore, given real constants $t_0$, $y_0$ and $y_1$, there is a unique solution to the initial value problem given by Eq. (4) and the initial conditions*

$$y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

# Variable coefficients second order linear ODE (Sect. 3.2).

Summary: The study the main properties of solutions to second order, linear, variable coefficients, ODE.

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel's theorem on the Wronskian.

# Review: Second order linear ODE.

## Definition

Given functions $a_1$, $a_0$, $b : \mathbb{R} \to \mathbb{R}$, the differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ given by

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is called a *second order linear* differential equation with *variable coefficients*.

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*then the linear combination $c_1 y_1(t) + c_2 y_2(t)$ is also a solution for any constants $c_1, c_2 \in \mathbb{R}$.*

# Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- **Existence and uniqueness of solutions.**
- Linearly dependent and independent functions.
- The Wronskian of two functions.
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# Existence and uniqueness of solutions.

## Theorem (Variable coefficients)

*If the functions* $a, b : (t_1, t_2) \to \mathbb{R}$ *are continuous, the constants* $t_0 \in (t_1, t_2)$ *and* $y_0$, $y_1 \in \mathbb{R}$, *then there exists a unique solution* $y : (t_1, t_2) \to \mathbb{R}$ *to the initial value problem*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

# Existence and uniqueness of solutions.

## Theorem (Variable coefficients)

*If the functions* $a, b : (t_1, t_2) \to \mathbb{R}$ *are continuous, the constants* $t_0 \in (t_1, t_2)$ *and* $y_0$, $y_1 \in \mathbb{R}$, *then there exists a unique solution* $y : (t_1, t_2) \to \mathbb{R}$ *to the initial value problem*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_1.$$

Remarks:

- Unlike the first order linear ODE where we have an explicit expression for the solution, there is no explicit expression for the solution of second order linear ODE.

# Existence and uniqueness of solutions.

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- Two integrations must be done to find solutions to second order linear. Therefore, initial value problems with two initial conditions can have a unique solution.

# Existence and uniqueness of solutions.

## Example

Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem

$$(t-1)y'' - 3ty' + 4y = t(t-1), \qquad y(-2) = 2, \qquad y'(-2) = 1.$$

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$$I_1 = (-\infty, 1). \qquad \triangleleft$$

# Existence and uniqueness of solutions.

Remark: The rest of the class is dedicated to show:

If functions $y_1$, $y_2$ are not proportional to each other and they are solutions of the equation

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then any other solution to this equation is given by

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- State a more precise and general result.

# Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- **Linearly dependent and independent functions.**
- The Wronskian of two functions.
- General and fundamental solutions.
- Abel's theorem on the Wronskian.

# Linearly dependent and independent functions.

## Definition

Two continuous functions $y_1,\ y_2 : (t_1, t_2) \subset \mathbb{R} \to \mathbb{R}$ are called *linearly dependent, (ld),* on the interval $(t_1, t_2)$ iff there exists a constant $c$ such that for all $t \in I$ holds

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(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are ld.

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We conclude: The functions $y_1$ and $y_2$ are li. $\triangleleft$

# Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
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- **The Wronskian of two functions.**
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- An alternative notation is: $W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

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Find the Wronskian of the functions:

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We obtain $W_{y_1 y_2}(t) = \sin^2(t)$. $\triangleleft$

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*The continuously differentiable functions* $y_1$, $y_2 : (t_1, t_2) \to \mathbb{R}$ *are linearly dependent iff* $W_{y_1 y_2}(t) = 0$ *for all* $t \in (t_1, t_2)$.

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Remark: Importance of the Wronskian:

- Sometimes it is not simple to decide whether two functions are proportional to each other.
- The Wronskian is useful to study properties of solutions to ODE without having the explicit expressions of these solutions. (See Abel's Theorem later on.)

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## Example

Show whether the following two functions form a l.d. or l.i. set:

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Solution: Compute their Wronskian:

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# The Wronskian of two functions.

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We conclude $W_{y_1 y_2}(t) = 0$, so the functions $y_1$ and $y_2$ are ld. $\triangleleft$

# The Wronskian of two functions.

Theorem (Variable coefficients)

*Let $y_1$ and $y_2$ be continuously differentiable solutions of*

$$y'' + a_1(t)y' + a_0(t)y = 0, \tag{5}$$

*where $a_1$, $a_2$ are continuous functions. Then, the following statement holds: Every solution $y$ of Eq. (5) can be decomposed as*

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$

*for appropriate constants $c_1$, $c_2$ iff functions $y_1$ and $y_2$ are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.*

Proof: See the textbook and the lecture notes.

# Variable coefficients second order linear ODE (Sect. 3.2).

- Review: Second order linear ODE.
- Existence and uniqueness of solutions.
- Linearly dependent and independent functions.
- The Wronskian of two functions.
- **General and fundamental solutions.**
- Abel's theorem on the Wronskian.

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Two solutions $y_1$, $y_2$ of the homogeneous equation

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are called *fundamental solutions* iff the functions $y_1$, $y_2$ are linearly independent, that is, iff $W_{y_1 y_2} \neq 0$.

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**Definition**

Given any two fundamental solutions $y_1$, $y_2$, and arbitrary constants $c_1$, $c_2$, the function

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is called the *general solution* of Eq. (5).

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Show that $y_1 = \sqrt{t}$ and $y_2 = 1/t$ are fundamental solutions of

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# Abel's theorem on the Wronskian.

## Theorem (Abel)

*If* $a_1$, $a_0 : (t_1, t_2) \to \mathbb{R}$ *are continuous functions and* $y_1$, $y_2$ *are continuously differentiable solutions of the equation*

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*Therefore, for any* $t_0 \in (t_1, t_2)$, *the Wronskian* $W_{y_1 y_2}$ *is given by*

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Remarks: If the the Wronskian of two solutions vanishes at the initial time, then it vanishes at all times.

# Abel's theorem on the Wronskian.

## Example

Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

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so, the solution is $W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, \frac{t^2}{t_0^2}\, e^{(t-t_0)}$.

# Abel's theorem on the Wronskian.

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Find the Wronskian of two solutions of the equation

$$t^2\,y'' - t(t+2)\,y' + (t+2)\,y = 0, \qquad t > 0.$$

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Denoting $c = \big(W_{y_1 y_2}(t_0)/t_0^2\big)\, e^{-t_0}$, then $W_{y_1 y_2}(t) = c\, t^2 e^t$. $\triangleleft$

# Review 1 for Exam 1.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks, webwork.
- Exam covers:
  - Linear equations (2.1).
  - Separable equations (2.2).
  - Homogeneous equations (2.2).
  - Modeling (2.3).
  - Non-linear equations (2.4).
  - Bernoulli equation (2.4).
  - Exact equations (2.6).
  - Exact equations with integrating factors (2.6).

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- Exam problems will be: Solve this equation. We don't tell you if the equation is linear, separable, etc. You must find that out.
- If you know what type of equation is, then the equation is simple to solve.
- The difficult part in Exam 1 is to know what type of equation is the one you have to solve.

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   (Several manipulations: $y' = F(y/t)$.)

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5. Exact equations.
   (Check one equation: $N\, y' + M = 0$, and $\partial_t N = \partial_y M$.)
6. Exact equation with integrating factor.
   (Very complicated to check.)

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The equation is exact: $\partial_t N = \partial_y M$.

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Solution: Recall: $u^4 = \dfrac{t^4}{4} + \displaystyle\int te^t\,dt + c$. Integration by parts:

$$\left.\begin{aligned} f &= t, & g' &= e^t, \\ f' &= 1, & g &= e^t, \end{aligned}\right\} \quad \Rightarrow \quad \int te^t\,dt = te^t - \int e^t\,dt$$

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We conclude: $y^4(t) = \frac{t^4}{4} + (t-1)e^t + 5$. Implicit form.

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We conclude that the unique solution to the IVP is

$$y(t) = -\left[\frac{t^4}{4} + (t-1)e^t + 5\right]^{1/4}.$$

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This is a separable equation for $v$: $\displaystyle\int \frac{2v}{v^2 - 1}\,v'\,dt = \int \frac{1}{t}\,dt + c.$

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A water tank initially has $V_0 = 100$ liters of water with $Q_0$ grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates $r_i$ and $r_o$ such that:

(a) The tank water volume is constant.

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$$\frac{dQ}{dt} = -\frac{r}{V_0}\,Q(t)$$

# Review 1 Exam 1.

## Example

A water tank initially has $V_0 = 100$ liters of water with $Q_0$ grams of salt. At $t_0 = 0$ fresh water is poured into the tank. The salt in the tank is always well mixed. Find the rates $r_i$ and $r_o$ such that:

(a) The tank water volume is constant.

(b) The time to reduce the salt in the tank to one percent of the initial value is $t_1 = 25$ min.

Solution:

Part (a): Water volume constant implies $r_i = r_o = r$. Then $V'(t) = 0$, so $V(t) = V_0$.

Part (b): First find the salt in the tank $Q(t)$: $\dfrac{dQ}{dt} = r_i q_i - r_o q_o(t)$.
Incoming fresh water: $q_i = 0$. Mixing: $q_o(t) = Q(t)/V(t)$.

$$\frac{dQ}{dt} = -\frac{r}{V_0}\, Q(t) \quad \Rightarrow \quad Q(t) = Q_0\, e^{-rt/V_0}.$$

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# Review 1 Exam 1.

## Example

Find the solution $y$ to the IVP

$$y' = \frac{2}{t}\,y - \frac{\sin(t)}{t}\,y^2, \qquad y(2\pi) = 2\pi, \qquad t > 0.$$

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Solution: Not linear.

# Review 1 Exam 1.

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## Example

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We solve the linear equation with the integrating factor method.

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$$t^2\Big(v' + \frac{2}{t}\,v\Big) = t^2\,\frac{\sin(t)}{t}$$

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Integrating: $t^2\,v = \displaystyle\int t\sin(t)\,dt + c$. The right hand side can be computed integrating by parts,

$$\int t\sin(t)\,dt = -t\cos(t) + \int \cos(t)\,dt, \quad \begin{cases} f = t, & g' = \sin(t), \\ f' = 1, & g = -\cos(t). \end{cases}$$

# Review 1 Exam 1.

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The initial condition: $4\pi^2\frac{1}{2\pi} = -2\pi\cos(2\pi) + 0 + c,$

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$$y = \frac{t^2}{\sin(t) - t\cos(t) + 4\pi} \qquad \triangleleft$$

# Review 1 Exam 1.

## Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

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$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

Solution: We first verify if the equation is not exact.

$$N = \left(x^3 e^y + \frac{x}{y}\right)$$

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$$N = \left(x^3 e^y + \frac{x}{y}\right) \quad \Rightarrow \quad \partial_x N = 3x^2 e^y + \frac{1}{y}.$$

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## Example

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$$N = \left(x^3 e^y + \frac{x}{y}\right) \quad \Rightarrow \quad \partial_x N = 3x^2 e^y + \frac{1}{y}.$$

$$M = (2x^2 e^y + 1) = 0 \quad \Rightarrow \quad \partial_y M = 2x^2 e^y.$$

So the equation is not exact. We now compute

$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)}$$

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$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)} = \frac{-x^2 e^y - \frac{1}{y}}{x\left(x^2 e^y + \frac{1}{y}\right)}$$

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$$\frac{\partial_y M - \partial_x N}{N} = \frac{2x^2 e^y - \left(3x^2 e^y + \frac{1}{y}\right)}{\left(x^3 e^y + \frac{x}{y}\right)} = \frac{-x^2 e^y - \frac{1}{y}}{x\left(x^2 e^y + \frac{1}{y}\right)} = -\frac{1}{x}.$$

# Review 1 Exam 1.

## Example

Find the integrating factor that converts the equation below into an exact equation, where

$$\left(x^3 e^y + \frac{x}{y}\right) y' + (2x^2 e^y + 1) = 0.$$

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