

## Boundary Value Problems (Sect. 10.1).

- ▶ Two-point BVP.
- ▶ Example from physics.
- ▶ Comparison: IVP vs BVP.
- ▶ Existence, uniqueness of solutions to BVP.
- ▶ Particular case of BVP: Eigenvalue-eigenfunction problem.

### Two-point Boundary Value Problem.

#### Definition

A *two-point BVP* is the following: Given functions  $p$ ,  $q$ ,  $g$ , and constants  $x_1 < x_2$ ,  $y_1, y_2$ ,  $b_1, b_2$ ,  $\tilde{b}_1, \tilde{b}_2$ ,

find a function  $y$  solution of the differential equation

$$y'' + p(x) y' + q(x) y = g(x),$$

together with the extra, *boundary conditions*,

$$b_1 y(x_1) + b_2 y'(x_1) = y_1,$$

$$\tilde{b}_1 y(x_2) + \tilde{b}_2 y'(x_2) = y_2.$$

#### Remarks:

- ▶ Both  $y$  and  $y'$  might appear in the boundary condition, evaluated at the same point.
- ▶ In this notes we only study the case of constant coefficients,

$$y'' + a_1 y' + a_0 y = g(x).$$

## Two-point Boundary Value Problem.

### Example

Examples of BVP. Assume  $x_1 \neq x_2$ .

(1) Find  $y$  solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y(x_2) = y_2.$$

(2) Find  $y$  solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y'(x_1) = y_1, \quad y'(x_2) = y_2.$$

(3) Find  $y$  solution of

$$y'' + a_1 y' + a_0 y = g(x), \quad y(x_1) = y_1, \quad y'(x_2) = y_2.$$

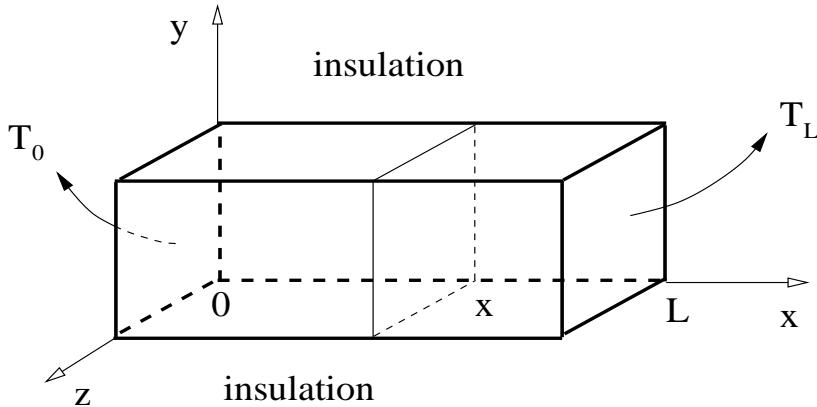
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## Example from physics.

**Problem:** The equilibrium (time independent) temperature of a bar of length  $L$  with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures  $T_0$ ,  $T_L$  is the solution of the BVP:

$$T''(x) = 0, \quad x \in (0, L), \quad T(0) = T_0, \quad T(L) = T_L,$$



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## Comparison: IVP vs BVP.

### Review: IVP:

Find the function values  $y(t)$  solutions of the differential equation

$$y'' + a_1 y' + a_0 y = g(t),$$

together with the initial conditions

$$y(t_0) = y_1, \quad y'(t_0) = y_2.$$

**Remark:** In physics:

- ▶  $y(t)$ : Position at time  $t$ .
- ▶ **Initial conditions:** Position and velocity at the initial time  $t_0$ .

## Comparison: IVP vs BVP.

### Review: BVP:

Find the function values  $y(x)$  solutions of the differential equation

$$y'' + a_1 y' + a_0 y = g(x),$$

together with the initial conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2.$$

**Remark:** In physics:

- ▶  $y(x)$ : A physical quantity (temperature) at a position  $x$ .
- ▶ **Boundary conditions:** Conditions at the boundary of the object under study, where  $x_1 \neq x_2$ .

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### Existence, uniqueness of solutions to BVP.

Review: The initial value problem.

#### Theorem (IVP)

Consider the homogeneous initial value problem:

$$y'' + a_1 y' + a_0 y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1,$$

and let  $r_{\pm}$  be the roots of the characteristic polynomial

$$p(r) = r^2 + a_1 r + a_0.$$

If  $r_+ \neq r_-$ , real or complex, then for every choice of  $y_0, y_1$ , there exists a unique solution  $y$  to the initial value problem above.

Summary: The IVP above always has a unique solution, no matter what  $y_0$  and  $y_1$  we choose.

## Existence, uniqueness of solutions to BVP.

### Theorem (BVP)

Consider the homogeneous boundary value problem:

$$y'' + a_1 y' + a_0 y = 0, \quad y(0) = y_0, \quad y(L) = y_1,$$

and let  $r_{\pm}$  be the roots of the characteristic polynomial

$$p(r) = r^2 + a_1 r + a_0.$$

- (A) If  $r_+ \neq r_-$ , real, then for every choice of  $L \neq 0$  and  $y_0, y_1$ , there exists a unique solution  $y$  to the BVP above.
- (B) If  $r_{\pm} = \alpha \pm i\beta$ , with  $\beta \neq 0$ , and  $\alpha, \beta \in \mathbb{R}$ , then the solutions to the BVP above belong to one of these possibilities:
  - (1) There exists a unique solution.
  - (2) There exists no solution.
  - (3) There exist infinitely many solutions.

## Existence, uniqueness of solutions to BVP.

Proof of IVP: We study the case  $r_+ \neq r_-$ . The general solution is

$$y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}, \quad c_1, c_2 \in \mathbb{R}.$$

The initial conditions determine  $c_1$  and  $c_2$  as follows:

$$y_0 = y(t_0) = c_1 e^{r_- t_0} + c_2 e^{r_+ t_0}$$

$$y_1 = y'(t_0) = c_1 r_- e^{r_- t_0} + c_2 r_+ e^{r_+ t_0}$$

Using matrix notation,

$$\begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution  $c_1$  and  $c_2$  for every constants  $y_0$  and  $y_1$  iff the  $\det(Z) \neq 0$ , where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

## Existence, uniqueness of solutions to BVP.

Proof of IVP:

$$\text{Recall: } Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_- e^{r_- t_0} & r_+ e^{r_+ t_0} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

A simple calculation shows

$$\det(Z) = (r_+ - r_-) e^{(r_+ + r_-) t_0} \neq 0 \Leftrightarrow r_+ \neq r_-.$$

Since  $r_+ \neq r_-$ , the matrix  $Z$  is invertible and so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Z^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

We conclude that for every choice of  $y_0$  and  $y_1$ , there exist a unique value of  $c_1$  and  $c_2$ , so the IVP above has a unique solution.  $\square$

## Existence, uniqueness of solutions to BVP.

Proof of BVP: The general solution is

$$y(x) = c_1 e^{r_- x} + c_2 e^{r_+ x}, \quad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions determine  $c_1$  and  $c_2$  as follows:

$$y_0 = y(0) = c_1 + c_2.$$

$$y_1 = y(L) = c_1 e^{r_- L} + c_2 e^{r_+ L}$$

Using matrix notation,

$$\begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution  $c_1$  and  $c_2$  for every constants  $y_0$  and  $y_1$  iff the  $\det(Z) \neq 0$ , where

$$Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

## Existence, uniqueness of solutions to BVP.

Proof of IVP: Recall:  $Z = \begin{bmatrix} 1 & 1 \\ e^{r_- L} & e^{r_+ L} \end{bmatrix} \Rightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ .

A simple calculation shows

$$\det(Z) = e^{r_+ L} - e^{r_- L} \neq 0 \Leftrightarrow e^{r_+ L} \neq e^{r_- L}.$$

(A) If  $r_+ \neq r_-$  and real-valued, then  $\det(Z) \neq 0$ .

We conclude: For every choice of  $y_0$  and  $y_1$ , there exist a unique value of  $c_1$  and  $c_2$ , so the BVP in (A) above has a unique solution.

(B) If  $r_{\pm} = \alpha \pm i\beta$ , with  $\alpha, \beta \in \mathbb{R}$  and  $\beta \neq 0$ , then

$$\det(Z) = e^{\alpha L} (e^{i\beta L} - e^{-i\beta L}) \Rightarrow \det(Z) = 2i e^{\alpha L} \sin(\beta L).$$

Since  $\det(Z) = 0$  iff  $\beta L = n\pi$ , with  $n$  integer,

- (1) If  $\beta L \neq n\pi$ , then BVP has a unique solution.
- (2) If  $\beta L = n\pi$  then BVP either has no solutions or it has infinitely many solutions.  $\square$

## Existence, uniqueness of solutions to BVP.

### Example

Find  $y$  solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = -1.$$

**Solution:** The characteristic polynomial is

$$p(r) = r^2 + 1 \Rightarrow r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad -1 = y(\pi) = -c_1 \Rightarrow c_1 = 1, \quad c_2 \text{ free.}$$

We conclude:  $y(x) = \cos(x) + c_2 \sin(x)$ , with  $c_2 \in \mathbb{R}$ .

The BVP has infinitely many solutions.  $\triangleleft$

## Existence, uniqueness of solutions to BVP.

### Example

Find  $y$  solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

**Solution:** The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 0 = y(\pi) = -c_1$$

The BVP has no solution. ◇

## Existence, uniqueness of solutions to BVP.

### Example

Find  $y$  solution of the BVP

$$y'' + y = 0, \quad y(0) = 1, \quad y(\pi/2) = 1.$$

**Solution:** The characteristic polynomial is

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i.$$

The general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x).$$

The boundary conditions are

$$1 = y(0) = c_1, \quad 1 = y(\pi/2) = c_2 \quad \Rightarrow \quad c_1 = c_2 = 1.$$

We conclude:  $y(x) = \cos(x) + \sin(x)$ .

The BVP has a unique solution. ◇

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### Particular case of BVP: Eigenvalue-eigenfunction problem.

Problem:

Find a number  $\lambda$  and a non-zero function  $y$  solutions to the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Remark:** This problem is similar to the eigenvalue-eigenvector problem in Linear Algebra: Given an  $n \times n$  matrix  $A$ , find  $\lambda$  and a non-zero  $n$ -vector  $\mathbf{v}$  solutions of

$$A\mathbf{v} - \lambda \mathbf{v} = \mathbf{0}.$$

Differences:

- ▶  $A \quad \longrightarrow \quad \left\{ \begin{array}{l} \text{computing a second derivative and} \\ \text{applying the boundary conditions.} \end{array} \right\}$
- ▶  $\mathbf{v} \quad \longrightarrow \quad \{\text{a function } y\}.$

## Particular case of BVP: Eigenvalue-eigenfunction problem.

### Example

Find every  $\lambda \in \mathbb{R}$  and non-zero functions  $y$  solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Remarks:** We will show that:

- (1) If  $\lambda \leq 0$ , then the BVP has no solution.
- (2) If  $\lambda > 0$ , then there exist infinitely many eigenvalues  $\lambda_n$  and eigenfunctions  $y_n$ , with  $n$  any positive integer, given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

- (3) Analogous results can be proven for the same equation but with different types of boundary conditions. For example, for  $y(0) = 0, y'(L) = 0$ ; or for  $y'(0) = 0, y'(L) = 0$ .

## Particular case of BVP: Eigenvalue-eigenfunction problem.

### Example

Find every  $\lambda \in \mathbb{R}$  and non-zero functions  $y$  solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Solution:** Case  $\lambda = 0$ . The equation is

$$y'' = 0 \quad \Rightarrow \quad y(x) = c_1 + c_2 x.$$

The boundary conditions imply

$$0 = y(0) = c_1, \quad 0 = c_1 + c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Since  $y = 0$ , there are NO non-zero solutions for  $\lambda = 0$ .

## Particular case of BVP: Eigenvalue-eigenfunction problem.

### Example

Find every  $\lambda \in \mathbb{R}$  and non-zero functions  $y$  solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Solution:** Case  $\lambda < 0$ . Introduce the notation  $\lambda = -\mu^2$ . The characteristic equation is

$$p(r) = r^2 - \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu.$$

The general solution is

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

The boundary condition are

$$0 = y(0) = c_1 + c_2,$$

$$0 = y(L) = c_1 e^{\mu L} + c_2 e^{-\mu L}.$$

## Particular case of BVP: Eigenvalue-eigenfunction problem.

### Example

Find every  $\lambda \in \mathbb{R}$  and non-zero functions  $y$  solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Solution:** Recall:  $y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$  and

$$c_1 + c_2 = 0, \quad c_1 e^{\mu L} + c_2 e^{-\mu L} = 0.$$

We need to solve the linear system

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow Z \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}$$

Since  $\det(Z) = e^{-\mu L} - e^{\mu L} \neq 0$  for  $L \neq 0$ , matrix  $Z$  is invertible, so the linear system above has a unique solution  $c_1 = 0$  and  $c_2 = 0$ .

Since  $y = 0$ , there are NO non-zero solutions for  $\lambda < 0$ .

## Particular case of BVP: Eigenvalue-eigenfunction problem.

### Example

Find every  $\lambda \in \mathbb{R}$  and non-zero functions  $y$  solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Solution:** Case  $\lambda > 0$ . Introduce the notation  $\lambda = \mu^2$ . The characteristic equation is

$$p(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm\mu i.$$

The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary condition are

$$0 = y(0) = c_1, \quad \Rightarrow \quad y(x) = c_2 \sin(\mu x).$$

$$0 = y(L) = c_2 \sin(\mu L), \quad c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu L) = 0.$$

## Particular case of BVP: Eigenvalue-eigenfunction problem.

### Example

Find every  $\lambda \in \mathbb{R}$  and non-zero functions  $y$  solutions of the BVP

$$y''(x) + \lambda y(x) = 0, \quad y(0) = 0, \quad y(L) = 0, \quad L > 0.$$

**Solution:** Recall:  $c_1 = 0$ ,  $c_2 \neq 0$ , and  $\sin(\mu L) = 0$ .

The non-zero solution condition is the reason for  $c_2 \neq 0$ . Hence

$$\sin(\mu L) = 0 \quad \Rightarrow \quad \mu_n L = n\pi \quad \Rightarrow \quad \mu_n = \frac{n\pi}{L}.$$

Recalling that  $\lambda_n = \mu_n^2$ , and choosing  $c_2 = 1$ , we conclude

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right). \quad \triangleleft$$