The Laplace Transform and the IVP (Sect. 6.2).

- Solving differential equations using $\mathcal{L}[\;]$.
  - Homogeneous IVP.
  - First, second, higher order equations.
  - Non-homogeneous IVP.
  - Recall: Partial fraction decompositions.

Solving differential equations using $\mathcal{L}[\;]$.

Remark: The method works with:

- Constant coefficient equations.
- Homogeneous and non-homogeneous equations.
- First, second, higher order equations.

Idea of the method:

$$\mathcal{L}\begin{bmatrix}\text{differential eq.}\\ \text{for } y(t).\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(2)}$$

$$\xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Using the table.)}\end{matrix}$$

Solving differential equations using $\mathcal{L}[\;]$.

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$$\xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Using the table.)}\end{matrix}$$

Recall:

(a) $\mathcal{L}[a\,f(t) + b\,g(t)] = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$;

(b) $\mathcal{L}\big[y^{(n)}\big] = s^n\,\mathcal{L}[y] - s^{(n-1)}\,y(0) - s^{(n-2)}\,y'(0) - \cdots - y^{(n-1)}(0)$.

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Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] \quad \Rightarrow \quad \mathcal{L}[y'' - y' - 2y] = 0.$$

The $\mathcal{L}[\ ]$ is a linear function, so

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0.$$

Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - \Big[s\,\mathcal{L}[y] - y(0)\Big] - 2\,\mathcal{L}[y] = 0,$$

We the obtain $(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0).$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Recall: $(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0).$

Differential equation for y $\xrightarrow{\mathcal{L}[\ ]}$ Algebraic equation for $\mathcal{L}[y]$.

Introduce the initial condition,

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1).$$

We can solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{(s-1)}{(s^2 - s - 2)}.$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-1)}{(s^2 - s - 2)}$.

The partial fraction method: Find the zeros of the denominator,

$$s^2 - s - 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} s_+ = 2, \\ s_- = -1, \end{cases}$$

Therefore, we rewrite: $\mathcal{L}[y] = \dfrac{(s-1)}{(s-2)(s+1)}$.

Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Recall: $\dfrac{(s-1)}{(s-2)(s+1)} = \dfrac{a}{s-2} + \dfrac{b}{s+1}$.

A simple calculation shows

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$$

$$(s-1) = s(a+b) + (a-2b) \quad \Rightarrow \quad \begin{cases} a + b = 1, \\ a - 2b = -1 \end{cases}$$

Hence, $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$. Then, $\mathcal{L}[y] = \dfrac{1}{3}\dfrac{1}{(s-2)} + \dfrac{2}{3}\dfrac{1}{(s+1)}$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \frac{1}{3}\frac{1}{(s-2)} + \frac{2}{3}\frac{1}{(s+1)}$. From the table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{s+1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3}\,\mathcal{L}[e^{2t}] + \frac{2}{3}\,\mathcal{L}[e^{-t}] = \mathcal{L}\Big[\frac{1}{3}\big(e^{2t} + 2e^{-t}\big)\Big]$$

We conclude that: $y(t) = \frac{1}{3}\big(e^{2t} + 2e^{-t}\big)$. $\triangleleft$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the $\mathcal{L}[\ ]$ of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$

The $\mathcal{L}[\ ]$ is a linear function,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = 0.$$

Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = 0,$$

Therefore, $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0)$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0)$.

Introduce the initial conditions, $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3$.

Solve for $\mathcal{L}[y]$ as follows: $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2 - 4s + 4)}$.

The partial fraction method: Find the roots of the denominator,

$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\big[4 \pm \sqrt{16-16}\big] \quad \Rightarrow \quad s_+ = s_- = 2.$$

We obtain: $\mathcal{L}[y] = \dfrac{(s-3)}{(s-2)^2}$.

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{(s-3)}{(s-2)^2}$.

This expression is already in the partial fraction decomposition.
Idea: Rewrite the right-hand side in terms of function in the table.

$$\mathcal{L}[y] = \frac{(s-2)+2-3}{(s-2)^2} = \frac{(s-2)}{(s-2)^2} - \frac{1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

From the Laplace transforms table:

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{(n+1)}} \quad \Rightarrow \quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

Homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{1}{s-2} - \dfrac{1}{(s-2)^2}$ and

$$\frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}] = \mathcal{L}\big[e^{2t} - te^{2t}\big].$$

We conclude that $y(t) = e^{2t} - te^{2t}$. $\triangleleft$

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- Solving differential equations using $\mathcal{L}[\ ]$.
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  - Recall: Partial fraction decompositions.

First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

Solution: Compute the $\mathcal{L}[\ ]$ of the equation,

$$\mathcal{L}\big[y^{(4)}\big] - 4\,\mathcal{L}[y] = 0.$$

$$\big[s^4\,\mathcal{L}[y] - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0)\big] - 4\,\mathcal{L}[y] = 0.$$

$$\big[s^4\,\mathcal{L}[y] - s^3 + 2s\big] - 4\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\,\mathcal{L}[y] = s^3 - 2s,$$

We obtain, $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

First, second, higher order equations.

Example

Use the Laplace Transform to find the solution of $y^{(4)} - 4y = 0$,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -2, \quad y'''(0) = 0.$$

Solution: Recall: $\mathcal{L}[y] = \dfrac{s^3 - 2s}{(s^4 - 4)}$.

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{s}{(s^2 + 2)}.$$

The last expression is in the table of Laplace Transforms,

$$\mathcal{L}[y] = \frac{s}{\left(s^2 + \big[\sqrt{2}\big]^2\right)} = \mathcal{L}\big[\cos(\sqrt{2}\,t)\big].$$

We conclude that $y(t) = \cos(\sqrt{2}\,t)$. $\triangleleft$

The Laplace Transform and the IVP (Sect. 6.2).

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Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Compute the Laplace transform of the equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\sin(2t)].$$

The right-hand side above can be expressed as follows,

$$\mathcal{L}[3\sin(2t)] = 3\,\mathcal{L}[\sin(2t)] = 3\,\frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.$$

Introduce this source term in the differential equation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \dfrac{6}{s^2+4}.$

Derivatives are transformed into power functions,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{6}{s^2+4}.$$

Rewrite the above equation,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0) + \frac{6}{s^2+4}.$$

Introduce the initial conditions,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \frac{6}{s^2+4}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \dfrac{6}{s^2+4}.$

Therefore, $\mathcal{L}[y] = \dfrac{(s-3)}{(s^2-4s+4)} + \dfrac{6}{(s^2-4+4)(s^2+4)}.$

From an Example above: $s^2 - 4s + 4 = (s-2)^2$,

$$\mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2} + \frac{6}{(s-2)^2(s^2+4)}.$$

From an Example above we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Recall: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \dfrac{6}{(s-2)^2(s^2+4)}$.

Use Partial fractions to simplify the last term above.

Find constants a, b, c, d, such that

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{as+b}{s^2+4} + \frac{c}{(s-2)} + \frac{d}{(s-2)^2}$$

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{(as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)}{(s^2+4)(s-2)^2}$$

$$6 = (as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4).$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: $6 = (as+b)(s-2)^2 + c(s-2)(s^2+4) + d(s^2+4)$.

$$6 = (as+b)(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2+4)$$

$$6 = a(s^3 - 4s^2 + 4s) + b(s^2 - 4s + 4) + c(s^3 + 4s - 2s^2 - 8) + d(s^2 + 4).$$

$$6 = (a+c)s^3 + (-4a + b - 2c + d)s^2 + (4a - 4b + 4c)s + (4b - 8c + 4d).$$

We obtain the system

$$a + c = 0, \qquad -4a + b - 2c + d = 0,$$

$$4a - 4b + 4c = 0, \qquad 4b - 8c + 4d = 6.$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: The solution for this linear system is

$$a = \frac{3}{8}, \qquad b = 0, \qquad c = -\frac{3}{8}, \qquad d = \frac{3}{4}.$$

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\frac{s}{s^2+4} - \frac{3}{8}\,\frac{1}{(s-2)} + \frac{3}{4}\,\frac{1}{(s-2)^2}.$$

Use the table of Laplace Transforms

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\mathcal{L}[\cos(2t)] - \frac{3}{8}\,\mathcal{L}[e^{2t}] + \frac{3}{4}\,\mathcal{L}[te^{2t}].$$

$$\frac{6}{(s-2)^2\,(s^2+4)} = \mathcal{L}\Big[\frac{3}{8}\,\cos(2t) - \frac{3}{8}\,e^{2t} + \frac{3}{4}\,te^{2t}\Big].$$

Non-homogeneous IVP.

Example

Use the Laplace transform to find the solution $y(t)$ to the IVP

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: Summary: $\mathcal{L}[y] = \mathcal{L}[e^{2t} - te^{2t}] + \dfrac{6}{(s-2)^2(s^2+4)}$,

$$\frac{6}{(s-2)^2\,(s^2+4)} = \mathcal{L}\Big[\frac{3}{8}\,\cos(2t) - \frac{3}{8}\,e^{2t} + \frac{3}{4}\,te^{2t}\Big].$$

$$\mathcal{L}[y(t)] = \mathcal{L}\Big[(1-t)\,e^{2t} + \frac{3}{8}\,(-1+2t)\,e^{2t} + \frac{3}{8}\,\cos(2t)\Big].$$

We conclude that

$$y(t) = (1-t)\,e^{2t} + \frac{3}{8}\,(2t-1)\,e^{2t} + \frac{3}{8}\,\cos(2t). \qquad \triangleleft$$