The integrating factor method (Sect. 2.1).

- Overview of differential equations.
- Linear Ordinary Differential Equations.
- The integrating factor method.
  - Constant coefficients.
  - The Initial Value Problem.
  - Variable coefficients.

Read:

- The direction field. Example 2 in Section 1.1 in the Textbook.
- See direction field plotters in Internet. For example, see:
  http://math.rice.edu/~dfield/dfpp.html
  This link is given in our class webpage.

Overview of differential equations.

Definition

A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

Remark: There are two main types of differential equations:

- Ordinary Differential Equations (ODE): Derivatives with respect to only one variable appear in the equation.

  Example:
  Newton's second law of motion: $m\,\mathbf{a} = \mathbf{F}$.

- Partial differential Equations (PDE): Partial derivatives of two or more variables appear in the equation.

  Example:
  The wave equation for sound propagation in air.

Overview of differential equations.

Example

Newton's second law of motion is an ODE: The unknown is $\mathbf{x}(t)$, the particle position as function of time t and the equation is

$$\frac{d^2}{dt^2}\mathbf{x}(t) = \frac{1}{m}\,\mathbf{F}(t, \mathbf{x}(t)),$$

with m the particle mass and $\mathbf{F}$ the force acting on the particle.

Example

The wave equation is a PDE: The unknown is $u(t, x)$, a function that depends on two variables, and the equation is

$$\frac{\partial^2}{\partial t^2}u(t, x) = v^2\frac{\partial^2}{\partial x^2}u(t, x),$$

with v the wave speed. Sound propagation in air is described by a wave equation, where u represents the air pressure.

Overview of differential equations.

Remark: Differential equations are a central part in a physical description of nature:

- Classical Mechanics:
  - Newton's second law of motion. (ODE)
  - Lagrange's equations. (ODE)
- Electromagnetism:
  - Maxwell's equations. (PDE)
- Quantum Mechanics:
  - Schrödinger's equation. (PDE)
- General Relativity:
  - Einstein equation. (PDE)
- Quantum Electrodynamics:
  - The equations of QED. (PDE).

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Linear Ordinary Differential Equations

Remark: Given a function $y : \mathbb{R} \to \mathbb{R}$, we use the notation

$$y'(t) = \frac{dy}{dt}(t).$$

Definition

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, a *first order ODE* in the unknown function $y : \mathbb{R} \to \mathbb{R}$ is the equation

$$y'(t) = f(t, y(t)).$$

The first order ODE above is called *linear* iff there exist functions $a, b : \mathbb{R} \to \mathbb{R}$ such that $f(t, y) = -a(t)\,y + b(t)$. That is, f is linear on its argument y, hence a first order linear ODE is given by

$$y'(t) = -a(t)\,y(t) + b(t).$$

Linear Ordinary Differential Equations

Example

A first order linear ODE is given by

$$y'(t) = -2\,y(t) + 3.$$

In this case function $a(t) = -2$ and $b(t) = 3$. Since these function do not depend on t, the equation above is called of constant coefficients.

Example

A first order linear ODE is given by

$$y'(t) = -\frac{2}{t}\,y(t) + 4t.$$

In this case function $a(t) = -2/t$ and $b(t) = 4t$. Since these functions depend on t, the equation above is called of variable coefficients.

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The integrating factor method.

Remark: Solutions to first order linear ODE can be obtained using the integrating factor method.

Theorem (Constant coefficients)

Given constants $a, b \in \mathbb{R}$ with $a \neq 0$, the linear differential equation

$$y'(t) = -a\,y(t) + b$$

has infinitely many solutions, one for each value of $c \in \mathbb{R}$, given by

$$y(t) = c\,e^{-at} + \frac{b}{a}.$$

The integrating factor method.

Proof: Multiply the differential equation $y'(t) + a\,y(t) = b$ by a non-zero function μ, that is,

$$\mu(t)\,\big(y' + ay\big) = \mu(t)\,b.$$

Key idea: The non-zero function μ is called an integrating factor iff holds

$$\mu\,\big(y' + ay\big) = \big(\mu\,y\big)'.$$

Not every function μ satisfies the equation above. Let us find what are the solutions μ of the equation above. Notice that

$$\mu\,\big(y' + ay\big) = \big(\mu\,y\big)' \quad \Leftrightarrow \quad \mu\,y' + \mu a y = \mu'\,y + \mu\,y'$$

$$ay\mu = \mu'\,y \quad \Leftrightarrow \quad a\mu = \mu' \quad \Leftrightarrow \quad \frac{\mu'(t)}{\mu(t)} = a.$$

The integrating factor method.

Proof: Recall: $\dfrac{\mu'(t)}{\mu(t)} = a$. Therefore,

$$\big[\ln\big(\mu(t)\big)\big]' = a \quad \Leftrightarrow \quad \ln\big(\mu(t)\big) = at + c_0,$$

$$\mu(t) = e^{at+c_0} \quad \Leftrightarrow \quad \mu(t) = e^{at}\, e^{c_0}.$$

Choosing the solution with $c_0 = 0$ we obtain $\mu(t) = e^{at}$.
For that function μ holds that $\mu\,(y' + ay) = \big(\mu\, y\big)'$. Therefore, multiplying the ODE $y' + ay = b$ by $\mu = e^{at}$ we get

$$(\mu y)' = b\mu \quad \Leftrightarrow \quad \big(e^{at} y\big)' = be^{at} \quad \Leftrightarrow \quad e^{at} y = \int be^{at}\, dt + c$$

$$y(t)\, e^{at} = \frac{b}{a} e^{at} + c \quad \Leftrightarrow \quad y(t) = c\, e^{-at} + \frac{b}{a}. \qquad \square$$

The integrating factor method.

Example

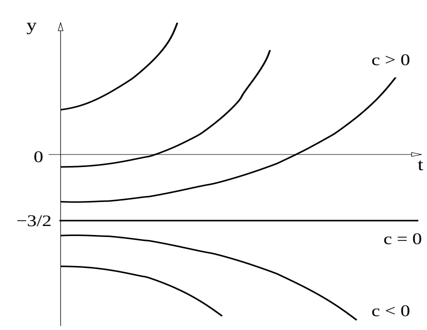
Find all functions y solution of the ODE $y' = 2y + 3$.

Solution: The ODE is $y' = -ay + b$ with $a = -2$ and $b = 3$.

The functions $y(t) = ce^{-at} + \dfrac{b}{a}$, with $c \in \mathbb{R}$, are solutions.

We conclude that the ODE has infinitely many solutions, given by

$$y(t) = c\, e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$



Since we did one integration, it is reasonable that the solution contains a constant of integration, $c \in \mathbb{R}$.

Verification: $c\, e^{2t} = y + (3/2)$, so $2c\, e^{2t} = y'$, therefore we conclude that y satisfies the ODE $y' = 2y + 3$. $\triangleleft$

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The Initial Value Problem.

Definition

The *Initial Value Problem* (IVP) for a linear ODE is the following: Given functions $a, b : \mathbb{R} \to \mathbb{R}$ and constants t_0, $y_0 \in R$, find a solution $y : \mathbb{R} \to \mathbb{R}$ of the problem

$$y' = a(t)\, y + b(t), \qquad y(t_0) = y_0.$$

Remark: The initial condition selects one solution of the ODE.

Theorem (Constant coefficients)

Given constants $a, b, t_0, y_0 \in \mathbb{R}$, with $a \neq 0$, the initial value problem

$$y' = -ay + b, \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \left(y_0 - \frac{b}{a}\right) e^{-a(t-t_0)} + \frac{b}{a}.$$

The Initial Value Problem.

Example

Find the solution to the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1.$$

Solution: Every solution of the ODE above is given by

$$y(t) = c\, e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 1$ selects only one solution:

$$1 = y(0) = c - \frac{3}{2} \quad \Rightarrow \quad c = \frac{5}{2}.$$

We conclude that $y(t) = \frac{5}{2}\, e^{2t} - \frac{3}{2}$. $\triangleleft$

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The integrating factor method.

Theorem (Variable coefficients)

Given continuous functions $a, b : \mathbb{R} \to \mathbb{R}$ and given constants $t_0, y_0 \in \mathbb{R}$, the IVP

$$y' = -a(t)y + b(t) \qquad y(t_0) = y_0$$

has the unique solution

$$y(t) = \frac{1}{\mu(t)}\Big[y_0 + \int_{t_0}^{t} \mu(s)b(s)ds\Big],$$

where the integrating factor function is given by

$$\mu(t) = e^{A(t)}, \qquad A(t) = \int_{t_0}^{t} a(s)ds.$$

Remark: See the proof in the Lecture Notes.

The integrating factor method.

Example

Find the solution y to the IVP

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: We first express the ODE as in the Theorem above,

$$y' = -\frac{2}{t}\,y + 4t.$$

Therefore, $a(t) = \dfrac{2}{t}$ and $b(t) = 4t$, and also $t_0 = 1$ and $y_0 = 2$.
We first compute the integrating factor function $\mu = e^{A(t)}$, where

$$A(t) = \int_{t_0}^{t} a(s)\,ds = \int_{1}^{t} \frac{2}{s}\,ds = 2\big[\ln(t) - \ln(1)\big]$$

$$A(t) = 2\ln(t) = \ln(t^2) \quad \Rightarrow \quad e^{A(t)} = t^2.$$

We conclude that $\mu(t) = t^2$.

The integrating factor method.

Example

Find the solution y to the IVP

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: The integrating factor is $\mu(t) = t^2$. Hence,

$$t^2\Big(y' + \frac{2}{t}\,y\Big) = t^2(4t) \quad \Leftrightarrow \quad t^2\,y' + 2t\,y = 4t^3$$

$$\big(t^2 y\big)' = 4t^3 \quad \Leftrightarrow \quad t^2 y = t^4 + c \quad \Leftrightarrow \quad y = t^2 + \frac{c}{t^2}.$$

The initial condition implies $2 = y(1) = 1 + c$, that is, $c = 1$.

We conclude that $y(t) = t^2 + \dfrac{1}{t^2}$. $\lhd$