

# Separable ODE.

### Definition

Given functions  $h, g : \mathbb{R} \to \mathbb{R}$ , a first order ODE on the unknown function  $y : \mathbb{R} \to \mathbb{R}$  is called *separable* iff the ODE has the form

$$h(y)\,y'(t)=g(t).$$

Remark:

A differential equation y'(t) = f(t, y(t)) is separable iff

$$y' = rac{g(t)}{h(y)} \quad \Leftrightarrow \quad f(t,y) = rac{g(t)}{h(y)}.$$

Notation:

In lecture: t, y(t) and h(y) y'(t) = g(t). In textbook: x, y(x) and M(x) + N(y) y'(x) = 0. Therefore: h(y) = N(y) and g(t) = -M(t).

# Separable ODE.

### Example

Determine whether the differential equation below is separable,

$$y'(t)=\frac{t^2}{1-y^2(t)}$$

Solution: The differential equation is separable, since it is equivalent to

$$\left(1-y^2\right)y'(t)=t^2 \quad \Rightarrow \quad \begin{cases} g(t)=t^2,\\ h(y)=1-y^2. \end{cases}$$

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Remark: The functions g and h are not uniquely defined. Another choice here is:

$$g(t)=c\ t^2,\quad h(y)=c\ (1-y^2),\quad c\in\mathbb{R}.$$

# Separable ODE.

### Example

Determine whether The differential equation below is separable,

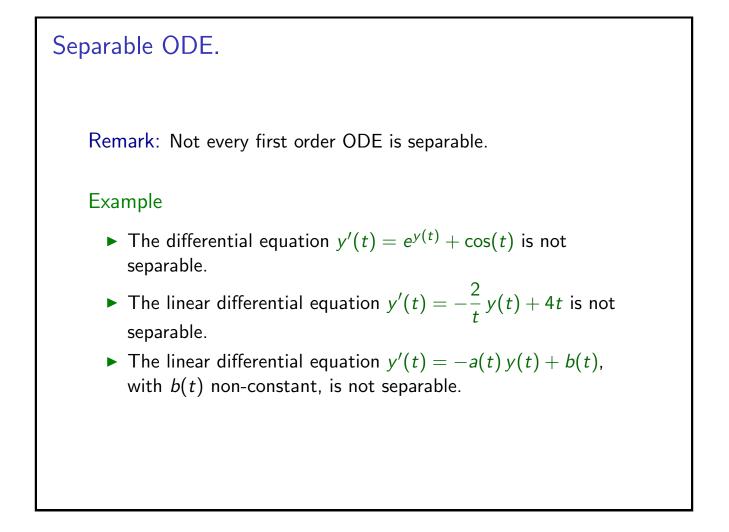
$$y'(t)+y^2(t)\,\cos(2t)=0$$

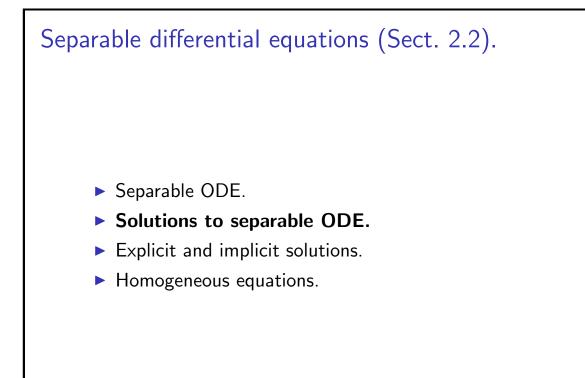
Solution: The differential equation is separable, since it is equivalent to

$$rac{1}{y^2} \, y'(t) = -\cos(2t) \quad \Rightarrow \quad \left\{ egin{array}{l} g(t) = -\cos(2t), \ h(y) = rac{1}{y^2}. \end{array} 
ight.$$

Remark: The functions g and h are not uniquely defined. Another choice here is:

$$g(t) = \cos(2t), \quad h(y) = -\frac{1}{y^2}$$





## Solutions to separable ODE.

Theorem (Separable equations)

If the functions  $g, h : \mathbb{R} \to \mathbb{R}$  are continuous, with  $h \neq 0$  and with primitives G and H, respectively; that is,

$$G'(t) = g(t), \qquad H'(u) = h(u),$$

then, the separable ODE

$$h(y)\,y'=g(t)$$

has infinitely many solutions  $y : \mathbb{R} \to \mathbb{R}$  satisfying the algebraic equation

$$H(y(t)) = G(t) + c,$$

where  $c \in \mathbb{R}$  is arbitrary.

Remark: Given functions g, h, find their primitives G, H.

## Solutions to separable ODE.

### Example

Find all solutions  $y : \mathbb{R} \to \mathbb{R}$  to the ODE  $y'(t) = \frac{t^2}{1 - y^2(t)}$ .

Solution: The equation is equivalent to  $(1 - y^2) y'(t) = t^2$ . Therefore, the functions g, h are given by

$$g(t) = t^2, \qquad h(u) = 1 - u^2.$$

Their primitive functions, G and H, respectively, are given by

		$G(t)=\frac{t^3}{3},$
$h(u)=1-u^2$	$\Rightarrow$	$H(u)=u-\frac{u^3}{3}.$

Then, the Theorem above implies that the solution y satisfies the algebraic equation

$$y(t)-rac{y^3(t)}{3}=rac{t^3}{3}+c,\quad c\in\mathbb{R}.$$

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# Solutions to separable ODE.

### Remarks:

- The equation  $y(t) \frac{y^3(t)}{3} = \frac{t^3}{3} + c$  is algebraic in y, since there is no y' in the equation.
- Every function y satisfying the algebraic equation

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c,$$

is a solution of the differential equation above.

We now verify the previous statement: Differentiate on both sides with respect to t, that is,

$$y'(t) - 3\left(\frac{y^2(t)}{3}\right)y'(t) = 3\frac{t^2}{3} \quad \Rightarrow \quad (1-y^2)y' = t^2.$$

Separable differential equations (Sect. 2.2).

- Separable ODE.
- Solutions to separable ODE.
- Explicit and implicit solutions.
- Homogeneous equations.

## Explicit and implicit solutions.

### Remark:

The solution  $y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c$  is given in implicit form.

### Definition

Assume the notation in the Theorem above. The solution y of a separable ODE is given in *implicit form* iff function y is specified by

H(y(t)) = G(t) + c,

The solution y of a separable ODE is given in *explicit form* iff function H is invertible and y is specified by

$$y(t) = H^{-1}(G(t) + c).$$

# Explicit and implicit solutions.

### Example

Use the main idea in the proof of the Theorem above to find the solution of the  $\ensuremath{\mathsf{IVP}}$ 

$$y'(t) + y^2(t)\cos(2t) = 0, \qquad y(0) = 1.$$

Solution: The differential equation is separable, with

$$g(t) = -\cos(2t), \qquad h(y) = \frac{1}{y^2}.$$

The main idea in the proof of the Theorem above is this: integrate on both sides of the equation,

$$rac{y'(t)}{y^2(t)} = -\cos(2t) \quad \Leftrightarrow \quad \int rac{y'(t)}{y^2(t)} \, dt = -\int \cos(2t) \, dt + c.$$

The substitution u = y(t), du = y'(t) dt, implies that

$$\int \frac{du}{u^2} = -\int \cos(2t) \, dt + c \quad \Leftrightarrow \quad -\frac{1}{u} = -\frac{1}{2}\sin(2t) + c.$$

# Explicit and implicit solutions.

#### Example

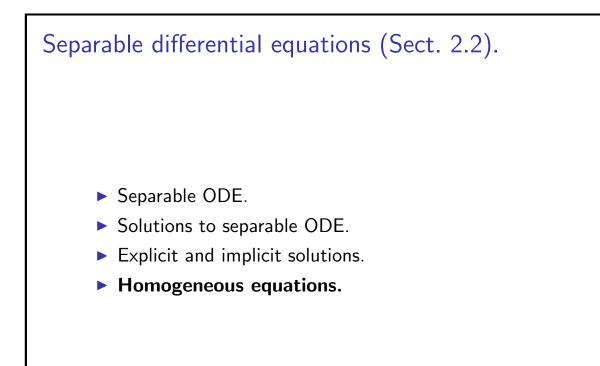
Use the main idea in the proof of the Theorem above to find the solution of the IVP

$$y'(t) + y^2(t)\cos(2t) = 0,$$
  $y(0) = 1.$ 

Solution: Recall:  $-\frac{1}{u} = -\frac{1}{2}\sin(2t) + c$ . Substitute the unknown function y back in the equation above,

 $-\frac{1}{y(t)} = -\frac{1}{2}\sin(2t) + c. \qquad \text{(Implicit form.)}$  $y(t) = \frac{2}{\sin(2t) - 2c}. \qquad \text{(Explicit form.)}$ 

The initial condition implies that  $1 = y(0) = \frac{2}{0 - 2c}$ , so c = -1. We conclude that  $y(t) = \frac{2}{\sin(2t) + 2}$ .



### Definition

The first order ODE y'(t) = f(t, y(t)) is called *homogeneous* iff for every numbers  $c, t, u \in \mathbb{R}$  the function f satisfies

f(ct, cu) = f(t, u).

Remark:

- The function f is invariant under the change of scale of its arguments.
- If f(t, u) has the property above, it must depend only on u/t.
- ▶ So, there exists  $F : \mathbb{R} \to \mathbb{R}$  such that  $f(t, u) = F\left(\frac{u}{t}\right)$ .
- Therefore, a first order ODE is homogeneous iff it has the form

$$y'(t) = F\left(\frac{y(t)}{t}\right).$$

## Homogeneous equations.

### Example

Show that the equation below is homogeneous,

$$(t-y)y'-2y+3t+\frac{y^2}{t}=0.$$

Solution: Rewrite the equation in the standard form

$$(t-y)y'=2y-3t-\frac{y^2}{t}$$
  $\Rightarrow$   $y'=\frac{\left(2y-3t-\frac{y^2}{t}\right)}{(t-y)}.$ 

Divide numerator and denominator by t. We get,

$$y' = \frac{\left(2y - 3t - \frac{y^2}{t}\right)}{\left(t - y\right)} \frac{\left(\frac{1}{t}\right)}{\left(\frac{1}{t}\right)} \quad \Rightarrow \quad y' = \frac{2\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right)^2}{\left[1 - \left(\frac{y}{t}\right)\right]}.$$

### Example

Show that the equation below is homogeneous,

$$(t-y)y'-2y+3t+\frac{y^2}{t}=0.$$

Solution: Recall:  $y' = \frac{2\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right)^2}{\left[1 - \left(\frac{y}{t}\right)\right]}.$ 

We conclude that the ODE is homogeneous, because the right-hand side of the equation above depends only on y/t.

Indeed, in our case:

$$f(t,y) = \frac{2y - 3t - (y^2/t)}{t - y}, \qquad F(x) = \frac{2x - 3 - x^2}{1 - x},$$

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and f(t, y) = F(y/t).

## Homogeneous equations.

### Example

Determine whether the equation below is homogeneous,

$$y'=\frac{t^2}{1-y^3}.$$

#### Solution:

Divide numerator and denominator by  $t^3$ , we obtain

$$y' = \frac{t^2}{(1-y^3)} \frac{\left(\frac{1}{t^3}\right)}{\left(\frac{1}{t^3}\right)} \quad \Rightarrow \quad y' = \frac{\left(\frac{1}{t}\right)}{\left(\frac{1}{t^3}\right) - \left(\frac{y}{t}\right)^3}$$

We conclude that the differential equation is not homogeneous.  $\lhd$ 

#### Theorem

If the differential equation y'(t) = f(t, y(t)) is homogeneous, then the differential equation for the unknown  $v(t) = \frac{y(t)}{t}$  is separable.

Remark: Homogeneous equations can be transformed into separable equations.

Proof: If y' = f(t, y) is homogeneous, then it can be written as y' = F(y/t) for some function F. Introduce v = y/t. This means,

 $y(t) = t v(t) \quad \Rightarrow \quad y'(t) = v(t) + t v'(t).$ 

Introducing all this into the ODE we get

$$v+t v'=F(v) \quad \Rightarrow \quad v'=rac{ig(F(v)-vig)}{t}.$$

This last equation is separable.

## Homogeneous equations.

### Example

Find all solutions y of the ODE  $y' = \frac{t^2 + 3y^2}{2ty}$ .

Solution: The equation is homogeneous, since

$$y' = \frac{t^2 + 3y^2}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Therefore, we introduce the change of unknown v = y/t, so y = t v and y' = v + t v'. Hence

$$v + t v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}$$

We obtain the separable equation  $v' = \frac{1}{t} \left( \frac{1+v^2}{2v} \right)$ .

### Example

Find all solutions y of the ODE  $y' = \frac{t^2 + 3y^2}{2ty}$ . Solution: Recall:  $v' = \frac{1}{t} \left( \frac{1+v^2}{2v} \right)$ . We rewrite and integrate it,  $\frac{2v}{1+v^2}v' = \frac{1}{t} \quad \Rightarrow \quad \int \frac{2v}{1+v^2}v'\,dt = \int \frac{1}{t}\,dt + c_0.$ The substitution  $u = 1 + v^2(t)$  implies du = 2v(t) v'(t) dt, so  $\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \quad \Rightarrow \quad \ln(u) = \ln(t) + c_0 \quad \Rightarrow \quad u = e^{\ln(t) + c_0}.$ But  $u = e^{\ln(t)}e^{c_0}$ , so denoting  $c_1 = e^{c_0}$ , then  $u = c_1t$ . Hence  $1+v^2=c_1t$   $\Rightarrow$   $1+\left(rac{y}{t}\right)^2=c_1t$   $\Rightarrow$   $y(t)=\pm t\sqrt{c_1t-1}.$