Name: key. ID Number: ____________

TA: Please, verify. Section Time: ____________

MTH 235
Exam 1 Makeup
February 4, 2010
50 minutes
Sects: 2.1-2.4, 2.6

No notes. No books. No Calculators.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. *Box your answers.*

1. (15 points) Find the integrating factor that converts the equation below for the unknown y into an exact equation, where

$$y' + t\,y\,y' + y^2 + \frac{y}{t} = 0.$$

You do not need to find the solution, only the integrating factor.

$$(1+ty)\,y' + \left(y^2 + \frac{y}{t}\right) = 0$$

$N = 1+ty \Rightarrow \partial_t N = y$

$M = y^2 + \frac{y}{t} \Rightarrow \partial_y M = 2y + \frac{1}{t}$

$$\frac{1}{N}(\partial_y M - \partial_t N) = \frac{1}{(1+ty)}\left(2y + \frac{1}{t} - y\right)$$

$$= \frac{1}{(1+ty)}\left(\frac{1}{t} + y\right)$$

$$= \frac{1}{t}\,\frac{1}{(1+ty)}(1+ty)$$

$$= \frac{1}{t} = \frac{\mu'}{\mu} \Rightarrow \ln\mu = \ln t$$

$$\boxed{\mu = t}$$

verify:

$(t + t^2 y)y' + ty^2 + y = 0$

$\tilde{N} = t + t^2 y \Rightarrow \partial_t \tilde{N} = 1 + 2ty$

$\tilde{M} = ty^2 + y \Rightarrow \partial_y \tilde{M} = 2ty + 1$ ✓

2. (17 points) Find all solutions y to the initial value problem

$$y' = -\frac{3}{t}y + \frac{\cos(\pi t)}{t^2}, \qquad y(1) = \frac{-1}{\pi^2}, \qquad t > 0.$$

linear eq.

$$y' + \frac{3}{t}y = \frac{\cos(\pi t)}{t^2}$$

$$a(t) = \frac{3}{t}, \qquad A(t) = 3\ln(t)$$
$$= \ln(t^3)$$

$$\mu(t) = e^{A(t)} \Rightarrow \boxed{\mu(t) = t^3}$$

$$t^3 y' + 3t^2 y = t\cos(\pi t)$$

$$t^3 y = c + \int t\cos(\pi t)\,dt \qquad \begin{array}{ll} f = t & g' = \cos(\pi t) \\ f' = 1 & g = \frac{1}{\pi}\sin(\pi t) \end{array}$$

$$t^3 y = c + \frac{t}{\pi}\sin(\pi t) - \int \frac{1}{\pi}\sin(\pi t)\,dt$$

$$\boxed{t^3 y = c + \frac{t}{\pi}\sin(\pi t) + \frac{\cos(\pi t)}{\pi^2}}$$

$$1^3\frac{(-1)}{\pi^2} = c + \frac{1}{\pi}\underbrace{\sin(\pi)}_{0} + \frac{\overbrace{\cos(\pi)}^{-1}}{\pi^2}$$

$$\frac{(-1)}{\pi^2} = c - \frac{1}{\pi^2} \Rightarrow \boxed{c = 0}$$

$$\boxed{y(t) = \frac{\sin(\pi t)}{\pi t^2} + \frac{\cos(\pi t)}{\pi^2 t^3}}$$

3. (17 points) A tank initially contains 200 of water with 50 lb of salt. The tank is rinsed with fresh water flowing in at a rate of 2 liters per minute and leaving the tank at the same rate. The water in the tank is well-stirred. Find the time such that the amount the salt in the tank is ~~10% the initial amount.~~ 5 lb.

$r_i = r_o = r = 2 \frac{l}{min}$ $\quad q_i = 0, \quad V_0 = 200\ l \quad Q_0 = 50$ lb

$V'(t) = 0 \Rightarrow \boxed{V(t) = V_0}$

$Q'(t) = r\,0 - \frac{r}{V_0} Q(t) \Rightarrow \boxed{Q(t) = Q_0 e^{-\frac{r}{V_0}t}}$

$5 \text{ lb} = \frac{Q_0}{10} = Q(t_1) = Q_0 e^{-\frac{r}{V_0}t_1}$

$\ln \frac{1}{10} = -\frac{r}{V_0} t_1$

$\ln 10 = \frac{r}{V_0} t_1 \Rightarrow \boxed{t_1 = \frac{V_0}{r} \ln 10}$

$\boxed{t_1 = 100 \ln 10}$

4. (17 points) Find all solutions y to the initial value problem

$$(y + t^2\, y)\, y' = 2t, \qquad y(0) = -2.$$

$(1+t^2)\, y\, y' = 2t$ Separable eq.

$$y\, y' = \frac{2t}{1+t^2}$$

$$\int y\, y'\, dt = \int \frac{2t}{1+t^2}\, dt + c$$

$u = y(t)$ $\qquad$ $v = 1+t^2$

$du = y'\, dt$ $\qquad$ $dv = 2t\, dt$

$$\int u\, du = \int \frac{dv}{v} + c$$

$$\frac{u^2}{2} = \ln(v) + c$$

$$\boxed{\frac{y^2}{2} = \ln(1+t^2) + c}$$

$$\frac{(-2)^2}{2} = \underbrace{\ln(1)}_{0} + c$$

$$\boxed{c = 2}$$

$$\boxed{y^2(t) = 2\ln(1+t^2) + 4}$$

$$\boxed{y(t) = -\sqrt{2\ln(1+t^2) + 4}}$$

implicit

5. (17 points) Find an ~~explicit~~ expression for all solutions y to the initial value problem

$$\cancel{y' = \frac{y-2t}{2y-t^2}}, \qquad y(1) = 0. \qquad y' = \frac{2y - t^2}{y - 2t}$$

Exact 2/3

$$(y-2t)\, y' + (t^2 - 2y) = 0$$

$$N = y - 2t \Rightarrow \partial_t N = -2$$

$$M = t^2 - 2y \Rightarrow \partial_y M = -2$$

$$\partial_y \psi = N \Rightarrow \partial_y \psi = y - 2t \Rightarrow \boxed{\psi = \frac{y^2}{2} - 2ty + g(t)}$$

$$\partial_t \psi = M$$

$$-2y + g' = \partial_t \psi = M = t^2 - 2y$$

$$g' = t^2 \Rightarrow \boxed{g(t) = \frac{t^3}{3}}$$

$$\boxed{\psi = \frac{y^2}{2} - 2ty + \frac{t^3}{3}}$$

$$\frac{y^2}{2} - 2ty + \frac{t^3}{3} = c \qquad y(1) = 0$$

$$0 - 0 + \frac{1}{3} = c$$

$$\boxed{\frac{y^2}{2} - 2ty + \frac{t^3}{3} = \frac{1}{3}}$$

6. (17 points) Find an explicit expression for all solutions y to the differential equation

$$t^2\, y' = t\, y - y^3.$$

Bernoulli eq.

$$y' = \frac{1}{t} y - \frac{1}{t^2} y^3$$

$$\frac{y'}{y^3} = \frac{1}{t}\frac{1}{y^2} - \frac{1}{t^2}$$

$$v = \frac{1}{y^2} \qquad v' = -2\frac{y'}{y^3}$$

$$-\frac{1}{2} v' = \frac{1}{t} v - \frac{1}{t^2}$$

$$v' + \frac{2}{t} v = \frac{2}{t^2}$$

$$t^2 v' + 2t\, v = 2$$

$$(t^2 v)' = 2$$

$$t^2 v = 2t + c$$

$$\boxed{v = \frac{2}{t} + \frac{c}{t^2}}$$

$$\frac{1}{y^2} = \frac{2}{t} + \frac{c}{t^2}$$

$$\frac{1}{y^2} = \frac{2t+c}{t^2}$$

$$y^2 = \frac{t^2}{2t+c}$$

$$\boxed{y(t) = \pm \frac{|t|}{\sqrt{2t+c}}}$$

#	Pts	Score
1	17	
2	17	
3	17	
4	17	
5	17	
6	15	
Σ	100	