40'

Name: key ID Number:

TA: Please Verify !! Section Time:

MTH 235
Exam 1 2
February ~~4~~, 2010
50 minutes
Sects: 2.1-2.4, 2.6

No notes. No books. No Calculators.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. *Box your answers.*

1. (15 points) Find the integrating factor that converts the equation below for the unknown y into an exact equation, where

$$4ty^3\,y' + 3t^2\,y' + 3ty + 4t^4 = 0.$$

You do not need to find the solution, only the integrating factor.

$$(4ty^3 + 3t^2)\,y' + (3ty + 4t^4) = 0$$

$N = 4t\,y^3 + 3t^2 \Rightarrow \partial_t N = 4y^3 + 6t$

$M = 3ty + 4t^4 \Rightarrow \partial_y M = 3t$

$\partial_t N \neq \partial_y M.$

$$\frac{1}{N}(\partial_y M - \partial_t N) = \frac{1}{(4ty^3+3t^2)}(3t - 4y^3 - 6t)$$

5 pts

$$= \frac{1}{t(4y^3+3t)}(-4y^3 - 3t)$$

5pts

$$\boxed{\frac{1}{N}(\partial_y M - \partial_t N) = -\frac{1}{t}} \Rightarrow \frac{\mu'}{\mu} = -\frac{1}{t} \Rightarrow \ln\mu = -\ln t \Rightarrow$$

$$\Rightarrow \ln\mu = \ln\frac{1}{t} \Rightarrow \boxed{\mu = \frac{1}{t}}$$

5 pts

$$(4y^3 + 3t)\, y' + (3y + 4t^3) = 0$$

$$\tilde{N} = 4y^3 + 3t \Rightarrow \partial_t \tilde{N} = 3$$

$$\tilde{M} = 3y + 4t^3 \Rightarrow \partial_y \tilde{M} = 3$$

$$\Rightarrow \boxed{\partial_t \tilde{N} = \partial_y \tilde{M}}$$

4

Not needed

2. (17 points) Find all solutions y to the initial value problem

$$t^2\,y' + 3t\,y = \cancel{4}\,5\,(t^2+1), \qquad y(1) = 2, \qquad t > 0.$$

Linear eq.

$$\boxed{y' + \frac{3}{t}\,y = 5\left(1 + \frac{1}{t}\right)}$$

2 pts

$$a(t) = \frac{3}{t} \qquad A(t) = \int \frac{3}{t}\,dt = 3\ln(t) \Rightarrow \boxed{A(t) = \ln(t^3)}$$

$$\mu = e^{A(t)} = e^{\ln(t^3)} \Rightarrow \boxed{\mu = t^3}$$

5 pts

$$\boxed{t^3 y' + 3t^2 y = 5(t^3 + t^2)}$$

$$(t^3 y)' = 5(t^3 + t^2)$$

$$t^3 y = 5\left(\frac{t^4}{4} + \frac{t^3}{3}\right) + c$$

$$\boxed{y(t) = \frac{5}{4}t + \frac{5}{3} + \frac{c}{t^3}}$$

7 pts

$$2 = y(1) = \frac{5}{4} + \frac{5}{3} + c \Rightarrow 2 = \frac{15+20}{12} + c \qquad c = \frac{24}{12} - \frac{35}{12}$$

$$\boxed{c = -\frac{11}{12}}$$

$$\boxed{y(t) = \frac{5}{4}t - \frac{11}{12}\,\frac{1}{t^3} + \frac{5}{3}}$$

3 pts

3. (17 points) A tank initially contains 90 liters of pure water. Water enters the tank at a rate of 3 liters per minute with a salt concentration of 2 grams per liter. The instantaneously mixed mixture leaves the tank at the same rate it enters the tank. Find the salt concentration in the tank at any time $t \geqslant 0$. Also find the limiting amount of salt in the tank in the limit $t \to \infty$.

$$V_0 = 90 \text{ l.}, \quad \boxed{r_i = 3 \frac{l}{min}} ; \quad q_i = 2 \frac{gr}{l.}$$

$$r_i = r_o = r \qquad q_o(t) = \frac{Q(t)}{V(t)} ; \qquad Q_0 = 0$$

$$V'(t) = r_i - r_o = 0 \Rightarrow \boxed{V(t) = V_0}$$

2 pts

$$Q'(t) = r\, q_i - r \frac{Q(t)}{V_0}$$

$$\boxed{Q'(t) + \frac{r}{V_0} Q(t) = r\, q_i}$$

5 pts

$$\mu = e^{\frac{r}{V_0} t}$$

$$\left(e^{\frac{r}{V_0} t} Q\right)' = r\, q_i\, e^{\frac{r}{V_0} t}$$

$$e^{\frac{r}{V_0} t} Q(t) - Q_0 = r\, q_i \frac{V_0}{r}\left(e^{\frac{r}{V_0} t} - 1\right)$$

$$e^{\frac{r}{V_0} t} Q(t) = Q_0 + q_i V_0 \left(e^{\frac{r}{V_0} t} - 1\right)$$

$$\boxed{Q(t) = (Q_0 - q_i V_0)\, e^{-\frac{r}{V_0} t} + q_i V_0}$$

$Q_0 = 0$ (Pure water)

$$\boxed{Q(t) = -\, q_i V_0\, e^{-\frac{r}{V_0} t} + q_i V_0}$$

7 pts

$$q_i V_0 = 2 \frac{g}{l} \, 90 \, l \Rightarrow \boxed{q_i V_0 = 180 \ g}$$

$$\frac{r}{V_0} = 3 \frac{l}{min} \frac{1}{90 \, l} \Rightarrow \boxed{\frac{r}{V_0} = \frac{1}{30} \frac{1}{min}}$$

$$\boxed{Q(t) = -180 \, e^{-t/30} + 180}$$

$$\boxed{\lim_{t \to \infty} Q(t) = q_i V_0} = 180 \ g$$

4 pts

4. (17 points) Find all solutions y to the equation

$$(t^2 + 2\,ty)\,y' = y^2.$$

Homogeneous eq.

$$y' = \frac{y^2}{t^2+2ty} \quad \frac{(1/t^2)}{(1/t^2)} \Rightarrow y' = \frac{\left(\frac{y}{t}\right)^2}{1+2\left(\frac{y}{t}\right)}$$

3pts

$$v = \frac{y}{t}, \quad y = tv \Rightarrow y' = v + tv'$$

$$tv' + v = \frac{v^2}{1+2v} \Rightarrow tv' = \frac{v^2}{1+2v} - v$$

$$= \frac{v^2 - v - 2v^2}{1+2v} \Rightarrow$$

$$\Rightarrow tv' = \frac{-v^2-v}{1+2v} \Rightarrow \boxed{\frac{(1+2v)\,v'}{v^2+v} = -\frac{1}{t}}$$

6pts

$$\int \frac{(1+2v)\,v'}{v^2+v}\,dt = -\int \frac{dt}{t} + c$$

$$u = v^2+v, \quad du = (2v+1)\,v'\,dt$$

$$\int \frac{du}{u} = -\int \frac{dt}{t} + c \Rightarrow \ln|u| = -\ln|t| + c$$

$$\ln|u| = -\ln|t| + c_0$$

$$= \ln\left(\frac{1}{|t|}\right) + c_0$$

$$\boxed{|u| = \frac{1}{|t|}c_1}, \quad c_1 = e^{c_0}$$

5pts

$$|v^2+v| = \frac{c_1}{|t|}, \quad v = \frac{y}{t}$$

$$\left|\frac{y^2}{t^2} + \frac{y}{t}\right| = \frac{c}{|t|}$$

$$\boxed{|y^2 + ty| = c|t|}$$

3pts

5. (17 points) Find an explicit expression for all solutions y to the initial value problem

$$y' = \frac{9t^2 + y - 1}{4y - t}, \qquad y(1) = 0.$$

$(4y - t)\, y' - (9t^2 + y - 1) = 0$

$N = 4y - t \quad \Rightarrow \quad \partial_t N = -1$

$M = -(9t^2 + y - 1) \quad \Rightarrow \quad \partial_y M = -1$

$\Rightarrow \partial_t N = \partial_y M.$

4pts

$\partial_y \psi = N \quad \Rightarrow \quad \partial_y \psi = 4y - t$

$\partial_t \psi = M$

$\boxed{\psi = 2y^2 - ty + g(t)}$

4pts

$-y + g'(t) = \partial_t \psi = M = -9t^2 - y + 1$

$g' = -9t^2 + 1 \quad \Rightarrow \quad \boxed{g(t) = -3t^3 + t}$

4pts

$\boxed{2y^2 - ty + t - 3t^3 = c}$

$y(1) = 0$

2pts

$0 - 0 + 1 - 3 = c \quad \Rightarrow \quad \boxed{c = -2}$

$\boxed{2y^2(t) - t\, y(t) + t - 3t^3 = -2}$

3pts

6. (17 points) Find an explicit expression for all solutions y to the differential equation

$$t^2 y' = 2t\,y - y^3.$$

$y' = \frac{2}{t} y - \frac{1}{t^2} y^3$ Bernoulli eq.

$\boxed{\frac{y'}{y^3} = \frac{2}{t} \frac{1}{y^2} - \frac{1}{t^2}}$, $\boxed{v = \frac{1}{y^2}}$, $\boxed{v' = -2 \frac{y'}{y^3}}$

$-\frac{1}{2} v' = \frac{2}{t} v - \frac{1}{t^2}$

5 pts

$v' + \frac{4}{t} v = \frac{2}{t^2}$

$a(t) = \frac{4}{t}$ $A(t) = \int \frac{4}{t}\, dt = 4 \ln(t) = \ln(t^4)$

4 pts

$\mu(t) = e^{\ln(t^4)} \Rightarrow \boxed{\mu = t^4}$

#	Pts	Score
1	17	
2	17	
3	17	
4	17	
5	17	
6	15	
Σ	100	

$t^4 v' + 4t^3 v = 2t^2$

5 pts

$(t^4 v)' = 2t^2$

$t^4 v = \frac{2}{3} t^3 + c$

$\boxed{v = \frac{2}{3} \frac{1}{t} + \frac{c}{t^4}}$

3 pts

$\frac{1}{y^2} = \frac{2t^3 + 3c}{3t^4}$

$\boxed{y(t) = \pm \sqrt{\frac{3t^4}{2t^3 + 3c}}}$