Practice Exam 3

(1)

(a)

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 2 \\ 2 & -8 & -3 \end{bmatrix} \begin{matrix} 1)\cdot -2 \\ 2) \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & -12 & -9 \end{bmatrix} \begin{matrix} 3) \\ 2) \end{matrix} \rightarrow \boxed{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = U}$$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}}$$

(b)

$$LU\underline{x} = \underline{b} \iff \begin{matrix} U\underline{x} = \underline{Y} \\ L\underline{Y} = \underline{b} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \underline{Y} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \Rightarrow \boxed{Y_1 = 1}$$

$$-1 + Y_2 = -2 \Rightarrow \boxed{Y_2 = -1}$$

$$2 + 3 + Y_3 = -1 \Rightarrow \boxed{Y_3 = -6}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \underline{X} = \begin{bmatrix} 1 \\ -1 \\ -6 \end{bmatrix} \Rightarrow \boxed{X_3 = -1}$$

$$4X_2 - 5 = -1 \Rightarrow \boxed{X_2 = 1}$$

$$X_1 + 2 - 3 = 1 \Rightarrow \boxed{X_1 = 2}$$

$$\boxed{\underline{X} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}$$

(2) (a) No: Take $p \in W$, with $\int_0^1 p \, dx < -1$.

Then $-p \notin W$, because $\int_0^1 (-p) \, dx > 1$

(b) Yes.

$p \in W \Rightarrow \int_0^1 x \, p \, dx = 0$

$q \in W \Rightarrow \int_0^1 x \, q \, dx = 0$

$$\int_0^1 x \, (a p + b q) \, dx = a \underbrace{\int_0^1 x \, p \, dx}_{=0} + b \underbrace{\int_0^1 x \, q \, dx}_{=0}$$

$$= 0 \Rightarrow \underline{(a p + b q) \in W.}$$

(3)
$$\left[\begin{array}{cccc|cccc} 1 & 2 & -4 & 3 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 2 & 0 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -4 & 3 & 1 & 0 & 0 & 0 \\ 0 & -3 & 9 & -3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 & 1 & 0 \\ 0 & -4 & 12 & -4 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -4 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -4 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ basis of $\mathbb{R}^4$ containing Basis of $\mathcal{N}(A)$

(4) $p(\lambda) = \det(A - \lambda I)$ characteristic Polynomial.

A diagonalizable $\Rightarrow$ $A = \mathbb{P} D \mathbb{P}^{-1}$

$D = \text{diag}[\lambda_1, \dots, \lambda_n]$, and $p(\lambda_i) = 0$ $i = 1, \dots, n$.

$$p(A) = p(\mathbb{P} D \mathbb{P}^{-1})$$

$$= \mathbb{P}\, p(D)\, \mathbb{P}^{-1}$$

$$= \mathbb{P}\, \text{diag}\,[\underset{=0}{p(\lambda_1)}, \dots, \underset{=0}{p(\lambda_n)}]\, \mathbb{P}^{-1}$$

$$= \mathbb{P}\, 0\, \mathbb{P}^{-1}$$

$$\boxed{p(A) = 0}$$

(5)(a) $T_{ss} = \left[[T(e_1)]_s , [T(e_2)]_s \right]$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{tikzcd} u \arrow[d, "P"'] & s \arrow[d, "Q"] \\ s & s \end{tikzcd} \quad T_{us}$$

$T_{us} = \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$

$P = I_{su}$, $Q = I_{ss}$

$T_{ss} = Q^{-1} T_{us} P$

$I_{us} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$P = \frac{1}{4-1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = I_{su}$

$T_{ss} = T_{us} \; I_{su}$

$= \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\boxed{T_{ss} = \frac{1}{3} \begin{bmatrix} -7 & 5 \\ -1 & 5 \end{bmatrix}}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{tikzcd} u \arrow[d, "P"'] & s \arrow[d, "Q"] \\ u & u \end{tikzcd}$$

(b) $T_{uu} = Q^{-1} T_{us} P$

$P = I_{uu}$, $Q = I_{us}$.

$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}$

$$\boxed{T_{uu} = \frac{1}{3} \begin{bmatrix} -7 & -1 \\ 5 & 5 \end{bmatrix}}$$

(6) (a)

$$A^T A = \begin{bmatrix}1&2&1\\2&-1&1\end{bmatrix}\begin{bmatrix}1&2\\2&-1\\1&1\end{bmatrix} = \begin{bmatrix}6&1\\1&6\end{bmatrix}$$

$$A^T \underline{b} = \begin{bmatrix}1&2&1\\2&-1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

$$\begin{bmatrix}6&1\\1&6\end{bmatrix}\underline{\hat{x}} = \begin{bmatrix}4\\2\end{bmatrix} \Rightarrow \underline{\hat{x}} = \frac{1}{36-1}\begin{bmatrix}6&-1\\-1&6\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}$$

$$\boxed{\underline{\hat{x}} = \frac{1}{35}\begin{bmatrix}22\\8\end{bmatrix}}$$

(b)

$$\underline{b}_\parallel = A\underline{\hat{x}} = \begin{bmatrix}1&2\\2&-1\\1&1\end{bmatrix}\frac{2}{35}\begin{bmatrix}11\\4\end{bmatrix} = \frac{2}{35}\begin{bmatrix}19\\18\\15\end{bmatrix}$$

$$\boxed{b_\parallel = \frac{2}{35}\begin{bmatrix}19\\18\\15\end{bmatrix}}$$ ✓

Check: $$\underline{b} - \underline{b}_\parallel = \frac{1}{35}\begin{bmatrix}35-38\\35-36\\35-30\end{bmatrix} = \frac{1}{35}\begin{bmatrix}-3\\-1\\5\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}-3\\-1\\5\end{bmatrix} = 0, \quad \begin{bmatrix}2\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}-3\\-1\\5\end{bmatrix} = 0$$ ✓

(7)

(a)

$$[-i\,,\,1\,,\,-1]\begin{bmatrix} w_1\\ w_2\\ w_3\end{bmatrix} = 0 = -i w_1 + w_2 - i w_3$$

$$i w_1 = w_2 - i w_3 \quad \Rightarrow \quad w_1 = -i w_2 - w_3$$

$$\underline{w} = \begin{bmatrix}-i\\ 1\\ 0\end{bmatrix} w_2 + \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix} w_3 \in W^{\perp}$$

$$\boxed{\Big\{ \underbrace{\begin{bmatrix}-i\\ 1\\ 0\end{bmatrix}}_{=\underline{w}_1}, \underbrace{\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}}_{=\underline{w}_2} \Big\} \text{ basis of } W^{\perp}}$$

(b)

$$[i\,,\,1\,,\,0]\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix} = -i \neq 0 \quad ; \quad \|\underline{w}_1\|^2 = [-i,1,0]\begin{bmatrix}i\\ 1\\ 0\end{bmatrix} = 1+1 = 2.$$

$$\underline{w}_{2\perp} = \underline{w}_2 - \frac{w_1 \cdot w_2}{\|w_1\|^2}\, w_1$$

$$= \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix} + \frac{i}{2}\begin{bmatrix}-i\\ 1\\ 0\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}-2\\ 0\\ 2\end{bmatrix} + \begin{bmatrix}1\\ i\\ 0\end{bmatrix} \right)$$

$$\underline{w}_{2\perp} = \frac{1}{2}\begin{bmatrix}-1\\ i\\ 2\end{bmatrix} \Rightarrow \boxed{\tilde{w}_{2\perp} = \begin{bmatrix}-1\\ i\\ 2\end{bmatrix}} \qquad \|\tilde{w}_{2\perp}\| = 1+1+4 = 6$$

$$\boxed{\Big\{ \frac{1}{\sqrt{2}}\begin{bmatrix}i\\ 1\\ 0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\ i\\ 2\end{bmatrix} \Big\}}$$ orthonormal basis of $W^{\perp}$

(8) $\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (\lambda-5)^2 - 16 = \lambda^2 - 10\lambda + 25 - 16$

(a)

$\lambda^2 - 10\lambda + 9 = 0 \Rightarrow \lambda = \dfrac{10 \pm \sqrt{100-36}}{2} = \dfrac{10 \pm \sqrt{64}}{2}$

$\lambda = \dfrac{10 \pm 8}{2} = 5 \pm 4 \Rightarrow \boxed{\lambda_+ = 9, \quad \lambda_- = 1}$

$\lambda_+ = 9$ $\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = v_2, \quad \boxed{\underline{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\lambda_- = 1, \quad \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2, \quad \boxed{\underline{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$$

(b) $X^2 = A = P \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}$, Propose $X = P\,\tilde{D}\,P^{-1}$

$\begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix} = D^2 = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$. $d_1 = \pm 3$, $d_2 = \pm 1$

$$\boxed{X_1 = P \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}; \quad X_2 = P \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}, \quad X_3 = P \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}, \quad X_4 = P \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}}$$

(9)
(a) $\begin{vmatrix} 8-\lambda & -18 \\ 3 & -7-\lambda \end{vmatrix} = (\lambda+7)(\lambda-8) + 54$

$= \lambda^2 - \lambda - 56 + 54$

$= \lambda^2 - \lambda - 2 = 0, \quad \lambda = \frac{1 \pm \sqrt{1+8}}{2}$

$\lambda_\pm = \frac{1 \pm 3}{2} \Rightarrow \boxed{\lambda_+ = 2} \quad \boxed{\lambda_- = -1}$

$\lambda_+ = 2. \quad \begin{bmatrix} 6 & -18 \\ 3 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 3v_3, \quad \boxed{\underline{v}_+ = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$

$\lambda_- = -1 \quad \begin{bmatrix} 9 & -18 \\ 3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \boxed{\underline{v}_- = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

(b) $e^{At} = P e^{Dt} P^{-1}; \quad P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$$\boxed{e^{At} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}}$$

(10) $\begin{vmatrix} -7-\lambda & 12 \\ -4 & 7-\lambda \end{vmatrix} = (\lambda-7)(\lambda+7) + 48$

$= \lambda^2 - 49 + 48 = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\boxed{\lambda_\pm = \pm 1}$

$\lambda_+ \quad \begin{bmatrix} -8 & 12 \\ -4 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow 2v_1 = 3v_2 \Rightarrow \boxed{\underline{v}_+ = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$

$\lambda_- \quad \begin{bmatrix} -6 & 12 \\ -4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = 2v_2 \Rightarrow \boxed{\underline{v}_- = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

$\underline{X}(t) = c_+ \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{t} + c_- \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{X}(0) = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} \Rightarrow \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{3-4} \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boxed{\underline{X}(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{t} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}}$