Practice Exam 1

(1)
$$A = \begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}, \quad \det(A) = 5\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 5(3) - 3(3) + (-3)$$

$$= 15 - 9 - 3 \Rightarrow \boxed{\det(A) = 3}$$

$$(A^{-1})_{21} = \frac{C_{12}}{\det(A)}$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3$$

$$(A^{-1})_{21} = \frac{(-3)}{3} \Rightarrow \boxed{(A^{-1})_{21} = -1}$$

$$(A^{-1})_{32} = \frac{C_{23}}{\det(A)}, \quad C_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= (-1)(5-6)$$

$$= 1$$

$$\boxed{(A^{-1})_{32} = \frac{1}{3}}$$

(2) (a) $\begin{vmatrix} 1 & 3 & k \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = (1)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$

$= 1 - 3(-1) + k(-4)$

$= 4 - 4k$

If Vol. $= 4 \Rightarrow 4 = 4 - 4k \Rightarrow \boxed{k = 0}$

(b) $\det(A) = 4 - 4k \quad ; \; k = 1 \Rightarrow$

$\Rightarrow \boxed{\det(A) = 0} \Rightarrow \boxed{A \text{ is Not injective}}$

$\dim N(A) + \dim R(A) = 3, \quad \dim N(A) \geq 1 \Rightarrow$

$3 = \dim N(A) + \dim R(A) \geq 1 + \dim R(A) \Rightarrow$

$\dim R(A) \leq 2 \Rightarrow \boxed{A \text{ is Not surjective}}$

(3)

$$\begin{bmatrix} -a+b \\ a-2b \\ a-7b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} a + \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} b. \Rightarrow$$

$$\Rightarrow V = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} \right\} \Leftrightarrow \boxed{V \text{ is a Subspace}}$$

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} = -1-2-7 = -10 \neq 0$$

$$\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \|\underline{v}_1\|^2 = 3; \quad \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}.$$

$$\underline{v}_{2\perp} = \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} - \frac{(-10)}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\left(\begin{bmatrix} 3 \\ -6 \\ -21 \end{bmatrix} + \begin{bmatrix} -10 \\ 10 \\ 10 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} -7 \\ 4 \\ -11 \end{bmatrix}$$

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 4 \\ -11 \end{bmatrix} \right\}} \quad \boxed{\text{orthogonal basis of } V.}$$

(4) (a) False (The false part is "for every $\underline{b} \in \mathbb{F}^m$")

Example $m=3,\ n=2$ $A \in \mathbb{R}^{3,2}$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{choose} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\boxed{A\underline{x} = \underline{b} \text{ has No solution.}}$$

(b) True : The maximum number of vectors forming a l. i. set in $\mathbb{R}^5$ is 5.

Seven vectors in $\mathbb{R}^5$ always form a l. d. set.

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(5) $T_{ss} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$T_{bb} = P^{-1} \, T_{ss} \, P \quad , \quad \boxed{P = I_{bs} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}}$

$P^{-1} = \frac{1}{-2-2} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}$

$T_{bb} = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

$\boxed{P^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}}$

$= \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 8 & -4 \end{bmatrix}$

$= \frac{1}{4} \begin{bmatrix} 18 & -10 \\ 2 & -2 \end{bmatrix}$

$$\boxed{T_{bb} = \frac{1}{2} \begin{bmatrix} 9 & -5 \\ 1 & -1 \end{bmatrix}}$$

(6) (a) $T_{S_3 S_2} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$ $S_{S_3 S_3} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $\boxed{[T \circ S]_{S_3 S_2} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}} = \boxed{\begin{bmatrix} 1 & -2 & 3 \\ 1 & 4 & -3 \end{bmatrix}}$

(c) $T \circ S : \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$\boxed{\dim N(T \circ S) = 1}$ $\left(= 3 - \operatorname{rank}(T \circ S) \right)$

so $\boxed{T \circ S \text{ is NOT injective}}$

$\operatorname{rank}(T \circ S) = 2 = \dim \mathbb{R}^2$

$\boxed{T \text{ is surjective}}$

(7) (a)

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$

$$A^T \underline{b} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \hat{\underline{x}} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \Rightarrow \hat{\underline{x}} = \frac{1}{36-25} \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 30-25 \\ -25+30 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \boxed{\hat{\underline{x}} = \frac{5}{11} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

(b)

$$A\hat{\underline{x}} - \underline{b} = \frac{5}{11} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{5}{11} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} - \frac{1}{11} \begin{bmatrix} 22 \\ 11 \\ 11 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} -12 \\ 4 \\ 4 \end{bmatrix}$$

$$\boxed{A\hat{\underline{x}} - \underline{b} = \frac{4}{11} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}}$$

$$\boxed{\begin{matrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0 \\ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 0 \end{matrix}}$$

$A\hat{x} - b \in R(A)^{\perp}$ ✓

(8) $A = \begin{bmatrix} -1/2 & -3 \\ 1/2 & 2 \end{bmatrix}$

$$\begin{vmatrix} -\frac{1}{2}-\lambda & -3 \\ \frac{1}{2} & 2-\lambda \end{vmatrix} = (\lambda-2)(\lambda+\tfrac{1}{2}) + \frac{3}{2}$$

$$= \lambda^2 - 2\lambda + \frac{1}{2}\lambda - 1 + \frac{3}{2}$$

$$= \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0 \Rightarrow$$

$$\Rightarrow \lambda_\pm = \frac{3/2 \pm \sqrt{\frac{9}{4} - \frac{4}{2}}}{2} = \frac{3/2 \pm \sqrt{\frac{1}{4}}}{2}$$

$$\lambda_\pm = \frac{3/2 \pm 1/2}{2} = \frac{3}{4} \pm \frac{1}{4} \Rightarrow \boxed{\begin{array}{c} \lambda_+ = 1 \\ \hline \lambda_- = \frac{1}{2} \end{array}}$$

$$\lambda_+ = 1, \quad \begin{bmatrix} -3/2 & -3 \\ \frac{1}{2} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -2v_2 \Rightarrow \boxed{v_+ = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

$$\lambda_- = \frac{1}{2}, \quad \begin{bmatrix} -1 & -3 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -3v_2 \Rightarrow \boxed{v_- = \begin{bmatrix} -3 \\ 1 \end{bmatrix}}$$

$$P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{-2+3} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$A = P D P^{-1} \quad , \quad \boxed{A = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}}$$

(b)

$$\lim_{k \to \infty} A^k = \lim_{k \to \infty} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1/2)^k \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$\boxed{\lim_{k \to \infty} A^k = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}}$$

(9)

(a) $\|\underline{v}\|^2 = \|3\underline{x} - \underline{y}\|^2 = \langle (3\underline{x} - \underline{y}), (3x, y) \rangle$

$= 9\,\|\underline{x}\|^2 + \|\underline{y}\|^2 + 3\,\underbrace{\langle \underline{x}, \underline{y} \rangle}_{=0 \; (\underline{x} \perp \underline{y})} - 3\,\underbrace{\langle \underline{y}, \underline{x} \rangle}_{=0}$

$= 9\,\underbrace{\|\underline{x}\|^2}_{=(1/3)^2} + \underbrace{\|\underline{y}\|^2}_{=1^2}$

$\|\underline{v}\|^2 = 1 + 1 = 2 \Rightarrow \boxed{\|\underline{v}\| = \sqrt{2}}$

(b) $T(v) = 3\,T(\underline{x}) - T(\underline{y}) = 6\underline{x} - 3\underline{y}$

$\|T(v)\|^2 = \|6\underline{x} - 3\underline{y}\|^2 = 36\,\|x\|^2 + 9\,\|\underline{y}\|^2$

$(\underline{x} \perp \underline{y})$

$\|T(v)\|^2 = 36\,\frac{1}{9} + 9 = 4 + 9 = 13$

$\boxed{\|T(v)\| = \sqrt{13}}$

(10)

$$\boxed{\lambda_1 = 2 \quad , \; r_1 = 2}$$

$$\boxed{\lambda_2 = 1 \quad , \; r_2 = 1.}$$

$\lambda_1 = 2$

$$A - 2I = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -1 & h \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & h-2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{h = 2}$$

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \left. \begin{array}{l} x_2 = -2x_3 \\ x_1, x_3 \text{ free.} \end{array} \right] \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ basis of } E_{\lambda=2}}$$