Name: ________________________________ **ID Number:** ____________________

MTH 415
Exam 3
April 1, 2009

No calculators or any other devices are allowed on this exam.
Read each question carefully. If any question is not clear, ask for clarification.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Answer each question completely, and show all your work.

1. (22 points) Consider the matrices $A = \begin{bmatrix} 1 & 3 & -3 \\ 2 & -2 & 10 \\ 3 & 1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & -3 \\ 0 & -6 & 6 \\ 1 & -1 & 3 \end{bmatrix}$.

(a) Is $N(A) = N(B)$? Justify your answer.
(b) Is $R(A) = R(B)$? Justify your answer.

Part (a): We now that $N(A) = N(B)$ iff $E_A = E_B$.

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 2 & -2 & 10 \\ 3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 \\ 0 & -8 & 16 \\ 0 & -8 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = E_A,$$

$$B = \begin{bmatrix} 1 & 5 & -3 \\ 0 & -6 & 6 \\ 1 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 \\ 0 & -6 & 6 \\ 0 & -6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = E_B.$$

Since $E_A \neq E_B$, we conclude $\boxed{N(A) \neq N(B)}$.

Part (b): We now that $R(A) = R(B)$ iff $E_{A^T} = E_{B^T}$.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ -3 & 10 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -8 \\ 0 & 16 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = E_{A^T},$$

$$B^T = \begin{bmatrix} 1 & 0 & 1 \\ 5 & -6 & -1 \\ -3 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -6 & -6 \\ 0 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = E_{B^T}.$$

Since $E_{A^T} = E_{B^T}$, we conclude $\boxed{R(A) = R(B)}$.

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2. (18 points) Find a basis and state the dimension of the vector space of all skew-symmetric 3×3 real matrices.

We are interested in finding a basis of the space of 3×3 skew-symmetric matrices, that is,

$$SS(3,3) = \{A \in M(3,3) \ : \ A = -A^T\}.$$

The condition $A = -A^T$ expressed in matrix components is $A_{ij} = -A_{ji}$, which implies that $A_{ii} = 0$, so an arbitrary element in the space $SS(3,3)$ is given by

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix} = A_{12} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + A_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + A_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Therefore, the set

$$\boxed{\mathcal{S} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}}$$

spans $SS(3,3)$. Since it is clear that the set above is linearly independent, this set is a basis for $SS(3,3)$. Since the basis has 3 elements, we conclude that $\boxed{\dim\big(SS(3,3)\big) = 3}$.

3. (20 points) Determine whether or not the set $\mathcal{U}$ is a basis for the subspace $W \subset \mathbb{R}^3$, where

$$\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}, \qquad W = \operatorname{Span}\left(\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\} \right).$$

First, notice that the set $\mathcal{U}$ is linearly independent, since the vector $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is not proportional to the vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. So, the set $\mathcal{U}$ is a basis of W iff $\operatorname{Span}(\mathcal{U}) = W$. Let us defined the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 4 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ 3 & 4 & 2 \end{bmatrix}.$$

Method 1: (Short.) Introduce the matrix $A = \begin{bmatrix} U^T \\ V^T \end{bmatrix}$, that is,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 4 \\ 1 & 3 & 2 \\ 2 & -2 & 4 \\ 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_A.$$

We have just shown that all the column vectors in V are linear combinations of the vectors in $\mathcal{U}$. Since we knew that $\mathcal{U}$ is linearly independent, we conclude that $\boxed{\mathcal{U} \text{ is a basis of } W}$.

Method 2: (Long.) From the definition of U and V we know that $R(U) = \operatorname{Span}(\mathcal{U})$ and $W = R(V)$, so we also know that:

$$U^T \xleftrightarrow{\text{row}} E_{U^T} \quad \Rightarrow \quad R(U) = R\big((E_{U^T})^T\big),$$
$$V^T \xleftrightarrow{\text{row}} E_{V^T} \quad \Rightarrow \quad R(V) = R\big((E_{V^T})^T\big).$$

We then conclude that $\operatorname{Span}(\mathcal{U}) = W$ iff $R(U) = R(V)$ iff $R\big((E_{U^T})^T\big) = R\big((E_{V^T})^T\big)$. A simple calculation shows:

$$U^T = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = E_{U^T},$$

$$V^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -2 & 4 \\ 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & -8 & 0 \\ 0 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_{V^T}.$$

Therefore,

$$R\big((E_{U^T})^T\big) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = R\big((E_{V^T})^T\big).$$

We then conclude that $R(U) = R(V)$, so $\boxed{\mathcal{U} \text{ is a basis of } W}$.

4. Let $\mathcal{A} = \{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ be two bases of $\mathbb{R}^2$ related by the equations

$$\boldsymbol{a}_1 = \boldsymbol{b}_1 + 2\boldsymbol{b}_2, \qquad \boldsymbol{a}_2 = -3\boldsymbol{b}_1 - 2\boldsymbol{b}_2.$$

Consider the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{A}}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}_{\mathcal{A}}$.

(a) (5 points) Find $[T]_{\mathcal{A}\mathcal{A}}$, that is, the matrix of the transformation T in the basis $\mathcal{A}$.
(b) (15 points) Find $[T]_{\mathcal{B}\mathcal{B}}$, that is, the the matrix of the operator T in the basis $\mathcal{B}$.

Part (a): We know that $[\boldsymbol{a}_1]_{\mathcal{A}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{A}}$ and $[\boldsymbol{a}_2]_{\mathcal{A}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{A}}$, then

$$T([\boldsymbol{a}_1]_{\mathcal{A}}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{A}}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{A}}, \qquad T([\boldsymbol{a}_2]_{\mathcal{A}}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{A}}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{A}}.$$

We conclude that

$$\boxed{[T]_{\mathcal{A}\mathcal{A}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{\mathcal{A}\mathcal{A}}}.$$

Part (b): We know that

$$[T]_{\mathcal{B}\mathcal{B}} = [I]_{\mathcal{A}\mathcal{B}}\,[T]_{\mathcal{A}\mathcal{A}}\,[I]_{\mathcal{B}\mathcal{A}}.$$

From the data of the problem it is simple to construct the matrix

$$[I]_{\mathcal{A}\mathcal{B}} = \big[[\boldsymbol{a}_1]_{\mathcal{B}}, [\boldsymbol{a}_2]_{\mathcal{B}}\big],$$

since

$$\boldsymbol{a}_1 = \boldsymbol{b}_1 + 2\boldsymbol{b}_2 \quad \Leftrightarrow \quad [\boldsymbol{a}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}},$$

$$\boldsymbol{a}_2 = -3\boldsymbol{b}_1 - 2\boldsymbol{b}_2 \quad \Leftrightarrow \quad [\boldsymbol{a}_2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}_{\mathcal{B}}.$$

Therefore,

$$[I]_{\mathcal{A}\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}_{\mathcal{A}\mathcal{B}} \quad \Rightarrow \quad [I]_{\mathcal{B}\mathcal{A}} = \big([I]_{\mathcal{A}\mathcal{B}}\big)^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 3 \\ -2 & 1 \end{bmatrix}_{\mathcal{B}\mathcal{A}}.$$

We then obtain,

$$[T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix}_{\mathcal{A}\mathcal{B}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{\mathcal{A}\mathcal{A}} \frac{1}{4}\begin{bmatrix} -2 & 3 \\ -2 & 1 \end{bmatrix}_{\mathcal{B}\mathcal{A}} \quad \Rightarrow \quad \boxed{[T]_{\mathcal{B}\mathcal{B}} = \frac{1}{2}\begin{bmatrix} 6 & -5 \\ 4 & -2 \end{bmatrix}}.$$

5. (20 points) Given the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, define the function the $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(\boldsymbol{x}, \boldsymbol{y}) = [\boldsymbol{x}]_{\mathcal{S}}^T A \, [\boldsymbol{y}]_{\mathcal{S}},$$

where $[\boldsymbol{x}]_{\mathcal{S}}$ and $[\boldsymbol{y}]_{\mathcal{S}}$ denote the components of the vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^2$ in the standard basis $\mathcal{S}$ of $\mathbb{R}^2$.

(a) Determine whether or not f defines an inner product in $\mathbb{R}^2$.

(b) Determine whether or not the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(\boldsymbol{x}) = \sqrt{f(\boldsymbol{x}, \boldsymbol{x})}$ defines a norm in $\mathbb{R}^2$.

Part (a): Let us write the function f in terms of the vector components,

$$f(\boldsymbol{x}, \boldsymbol{y}) = \big[x_1, x_2\big] \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2x_1y_1 + 2x_2y_2 + x_1y_2 - x_2y_1.$$

Since matrix A is not symmetric, this suggests that the symmetry property of an inner product could fail in the case of f. Let us start verifying this symmetry property. We already have computed $f(\boldsymbol{x}, \boldsymbol{y})$, let us now compute:

$$f(\boldsymbol{y}, \boldsymbol{x}) = 2y_1x_1 + 2y_2x_2 + y_1x_2 - y_2x_1$$

therefore,

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) - f(\boldsymbol{y}, \boldsymbol{x}) &= (2x_1y_1 + 2x_2y_2 + x_1y_2 - x_2y_1) - (2y_1x_1 + 2y_2x_2 + y_1x_2 - y_2x_1) \\ &= 2(x_1y_2 - x_2y_1) \\ &\neq 0 \qquad \text{for arbitrary } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2. \end{aligned}$$

We conclude that $\boxed{f \text{ does not define an inner product}}$.

Part (b): Let us write the function g in terms of the vector components,

$$g(\boldsymbol{x}) = \sqrt{f(\boldsymbol{x}, \boldsymbol{x})} = \sqrt{2(x_1)^2 + 2(x_2)^2} \quad \Rightarrow \quad g(\boldsymbol{x}) = \sqrt{2}\, \|\boldsymbol{x}\|_2,$$

where $\| \ \|_2$ denotes the 2-norm in $\mathbb{R}^2$. Since g is proportional to a norm with positive proportionality factor, then $\boxed{g \text{ defines a norm}}$.