Name: ________________________ ID Number: ________________

MTH 415
Exam 2
August 4, 2010
50 minutes
Sects: 3.1, 3.2, 4.1-4.4, 5.1-5.5.

One notes page, handwritten. No Calculator.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. $\boxed{\textit{Box your answers.}}$

1. (a) (12 points) Prove the following statement: If matrix $\mathsf{A} \in \mathbb{F}^{n,n}$ is skew-symmetric and the number n is odd, then matrix A is not invertible.

(b) (8 points) Is this result also true for n even? If "yes," prove it; if "no," show it with an example using 2×2 matrices.

(a) $A = -A^T \Rightarrow \det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A) \Rightarrow$

$\Rightarrow \det(A) = 0 \Rightarrow$ $\boxed{\text{A is Not invertible}}$

(b) $\boxed{A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \Rightarrow A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A.$

$\det(A) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 0 + 1 = 1$

$\boxed{\text{No}}$

2. (20 points) Determine whether the subsets W_1 and W_2 and are subspaces of $\mathbb{R}^{3,3}$. If your answer is "yes", then find a basis for the subspace.

(a) $W_1 = \left\{ \mathsf{A} \in \mathbb{R}^{3,3} \ : \ \mathsf{A} + \mathsf{A}^T = \mathsf{I}_3 \right\}$;

(b) $W_2 = \left\{ \mathsf{A} \in \mathbb{R}^{3,3} \ : \ \mathsf{A} - \mathsf{A}^T = 0 \quad \text{and} \quad \operatorname{tr}(\mathsf{A}) = 0 \right\}$.

(a) $0 + 0^T = 0 \neq I_3 \Rightarrow 0 \notin W_1$, $\boxed{W_1 \text{ is not a subspace.}}$

(b) $A \in W_2$, $\quad A = A^T$ and $tr(A) = 0$

$B \in W_2$, $\quad B = B^T$ and $tr(B) = 0$

For all $a, b \in \mathbb{R}$ holds:

$$(aA + bB)^T = a A^T + b B^T = (aA + bB)$$

$$tr(aA + bB) = a \underbrace{tr(A)}_{=0} + b \underbrace{tr(B)}_{=0} = 0$$

$\therefore$ $(aA + bB) \in W_2$.

$\boxed{W_2 \text{ is a subspace.}}$

Basis $\longrightarrow$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & -a_{11}-a_{22} \end{bmatrix}$$

$$= a_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ a_{13}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The set $\boxed{\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}}$

is l.i. and spans W_2. Thus $\mathcal{A}$ is a basis.

3. (20 points) Determine whether the set $\mathcal{U}$ is a basis for the subspace $W \subset \mathbb{R}^3$, where

$$\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} \qquad W = \operatorname{Span}\left(\left\{ \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix} \right\} \right).$$

Set $\mathcal{U}$ is l.i. since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not proportional to $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Find x_1, x_2, Sol. of $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

y_1, y_2 Sol. of $\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $\Rightarrow$

z_1, z_2 Sol. of $\begin{bmatrix} -2 \\ 4 \\ 10 \end{bmatrix} = z_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|ccc} 1 & 3 & 1 & 0 & -2 \\ 2 & 2 & -2 & 4 & 4 \\ 3 & 1 & -5 & 8 & 10 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 3 & 1 & 0 & -2 \\ 0 & -4 & -4 & 4 & 8 \\ 0 & -8 & -8 & 8 & 16 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 3 & 1 & 0 & -2 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -2 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & -2 & 3 & 4 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

All systems for x_1, x_2, y_1, y_2, z_1, z_2 are consistents, So,

$\boxed{\operatorname{Span} \mathcal{U} = W}$
$\boxed{\mathcal{U} \text{ is l.i.}}$ $\Rightarrow$ $\boxed{\mathcal{U} \text{ is a basis for } W}$

4. Consider the linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\left[T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_s\right)\right]_{\tilde{s}} = \begin{bmatrix} x_1 - x_2 - x_3 \\ -x_1 + 2x_2 + x_3 \end{bmatrix}_{\tilde{s}}, \qquad \left[S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_s\right)\right]_s = \begin{bmatrix} -x_1 + x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{bmatrix}_s,$$

where $\mathcal{S}$ is a standard basis of $\mathbb{R}^3$ and $\tilde{\mathcal{S}}$ is the standard basis of $\mathbb{R}^2$.

(a) (5 points) Find a matrix $[T]_{s\tilde{s}}$ and the matrix $[S]_{ss}$.

(b) (10 points) Is T injective? Is T surjective? Justify your answer.

(c) (5 points) Find the matrix of the composition $T \circ S : \mathbb{R}^3 \to \mathbb{R}^2$ in the standard basis, that is, find $[T \circ S]_{s\tilde{s}}$. Justify your answer.

(a) $[T]_{s\tilde{s}} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$, $[S]_{ss} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

(b) $[T]_{s\tilde{s}} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow$

$\Rightarrow \begin{cases} N(T) \neq \{\underline{0}\} \Rightarrow \boxed{\text{T Not injective}} \\ \left.\begin{array}{l} \dim R(T) = 2 = \dim \mathbb{R}^2 \\ R(T) \subset \mathbb{R}^2 \end{array}\right\} \Rightarrow R(T) = \mathbb{R}^2 \Rightarrow \boxed{T \text{ is surjective}} \end{cases}$

(c) $[T \circ S]_{s\tilde{s}} = [T]_{s\tilde{s}} [S]_{ss} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

$$\boxed{[T \circ S]_{s\tilde{s}} = \begin{bmatrix} -2 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}}$$

5. Consider the vector space $\mathbb{R}^2$ with ordered bases

$$\mathcal{S} = \left(\mathsf{e}_{1s} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_s, \mathsf{e}_{2s} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_s \right), \qquad \mathcal{U} = \left(\mathsf{u}_{1s} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_s, \mathsf{u}_{2s} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_s \right).$$

Let $\boldsymbol{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$[\boldsymbol{T}(\boldsymbol{u}_1)]_s = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_s, \quad [\boldsymbol{T}(\boldsymbol{u}_2)]_s = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_s.$$

(a) (5 points) Find the matrix T_{us}.

(b) (15 points) Find the matrix T_{su}.

(a.) $$T_{us} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$P \; u \to S \qquad S \xrightarrow{Q} u \qquad T_{us} \quad T_{su}$

$$T_{su} = Q^{-1} \, T_{us} \, P$$

$$P = I_{su}, \quad Q = I_{us}$$

$$Q^{-1} = I_{su}.$$

$$\Rightarrow \boxed{T_{su} = I_{su} \, T_{us} \, I_{su}}$$

$$I_{su} = [I_{us}]^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{(1-4)} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\boxed{I_{su} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}}$$

$$T_{su} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & 7 \\ 5 & -1 \end{bmatrix}$$

$$T_{su} = \frac{1}{9} \begin{bmatrix} 15 & -9 \\ -15 & 13 \end{bmatrix} \Rightarrow \boxed{T_{su} = \frac{1}{3} \begin{bmatrix} 5 & -3 \\ -5 & 5 \end{bmatrix}}$$

#	Pts	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Σ	100	