Name: ______________________________ ID Number: ____________________

MTH 415
Exam 1
July 21, 2010
50 minutes
Sects: 1.1-1.4, 2.1-2.6

No calculators or any other devices allowed.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. $\boxed{\textit{Box your answers.}}$

1. (15 points) Express matrix A as a sum of a symmetric matrix and a skew-symmetric matrix, where

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

$$S = \frac{A + A^T}{2} = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$$

$$\boxed{S = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}}$$

$$T = \frac{A - A^T}{2} = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

$$\boxed{T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}}$$

2. (15 points) Determine which of the following functions $T, S : \mathbb{R}^2 \to \mathbb{R}^2$ is linear:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ 3x_1x_2 \end{bmatrix}, \qquad S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix}.$$

If a function is linear, give a proof; if a function is not linear, show it with an example.

$$T(a\underline{x}) = T\left(\begin{bmatrix} a x_1 \\ a x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 a x_2 \\ 3 a x_1 a x_2 \end{bmatrix} = a \begin{bmatrix} 2x_2 \\ 3x_1x_2 a \end{bmatrix} \neq a\, T(\underline{x})$$

$\boxed{T \text{ is NOT linear}}$

$$S(a\underline{x} + b\underline{y}) = S\left(\begin{bmatrix} a x_1 + b y_1 \\ a x_2 + b y_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (a x_1 + b y_1) + (a x_2 + b y_2) \\ 3(a x_1 + b y_1) \end{bmatrix}$$

$$= \begin{bmatrix} a(x_1 + x_2) + b(y_1 + y_2) \\ a3x_1 + b3y_1 \end{bmatrix}$$

$$= a \begin{bmatrix} x_1 + x_2 \\ 3x_1 \end{bmatrix} + b \begin{bmatrix} y_1 + y_2 \\ 3y_1 \end{bmatrix}$$

$$= a\, S(\underline{x}) + b\, S(\underline{y})$$

$\boxed{S \text{ is lineer}}$

3. (20 points) Given the matrices $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find matrix X solution of the matrix equation

$$AXA - 6\,AX + 8\,AB = 0.$$

$A(XA - 6X) = -8\,AB$

$\det(A) = 9 - 1 = 8$

$A^{-1} = \frac{1}{8}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$XA - 6X = -8B$

$X(A - 6I) = -8B$

$A - 6I = \begin{bmatrix} 3-6 & -1 \\ -1 & 3-6 \end{bmatrix} = -\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$(A - 6I)^{-1} = \left(+\frac{1}{8}\right)\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$

$X = -8B(A - 6I)^{-1}$

$= -8\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(+\frac{1}{8}\right)\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$

$= -\begin{bmatrix} -1 & -5 \\ -5 & -9 \end{bmatrix}$

$$\boxed{X = \begin{bmatrix} 1 & 5 \\ 5 & 9 \end{bmatrix}}$$

4. (20 points) Consider the matrices

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -3 & 2 & 7 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix}.$$

Verify whether the following equations hold:

$$N(\mathsf{A}) = N(\mathsf{B})?, \quad N(\mathsf{A}^T) = N(\mathsf{B}^T)?, \quad R(\mathsf{A}) = R(\mathsf{B})?, \quad R(\mathsf{A}^T) = R(\mathsf{B}^T)?.$$

Justify your answers.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -3 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 8 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = E_A$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_B$$

$E_B \neq E_A \Rightarrow$ $\boxed{\begin{matrix} N(A) \neq N(B) \\ R(A^T) \neq R(B^T) \end{matrix}}$

$$A^T = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 2 & 2 \\ 3 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 \\ 0 & -4 & 8 \\ 0 & -8 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = E_{A^T}$$

$$B^T = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = E_{B^T}$$

$E_{B^T} = E_{A^T} \Rightarrow$ $\boxed{\begin{matrix} N(A^T) = N(B^T) \\ R(A) = R(B) \end{matrix}}$

5. (15 points) Determine whether the following statement true or false:

"Given a matrix $\mathsf{A} \in \mathbb{F}^{m,n}$, it holds that $\operatorname{tr}(\mathsf{A}^T\mathsf{A}) = 0$ if and only if $\mathsf{A} = 0$."

If the statement is true, prove it; if the statement is false, give an example showing it.

If $A \in \mathbb{R}^{m,n}$

$$tr(A^T A) = \sum_{i=1}^{n} (A^T A)_{ii}$$

$$\boxed{(A^T A)_{ij} = \sum_{k=1}^{n} (A^T)_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj}}$$

$$tr(A^T A) = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ki} A_{ki}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} (A_{ki})^2 = 0 \iff \boxed{A_{ki} = 0}$$

$$\iff \boxed{A = 0}$$

$\boxed{\text{True}}$ $\boxed{\text{in } \mathbb{R}^{m,n}}$

If $A \in \mathbb{C}^{m,n}$, then it is $\boxed{\text{False.}}$ $\boxed{\text{in } \mathbb{C}^{m,n}}$

Example. $A \in \mathbb{C}^{2,2}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$A^T A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & * \\ * & b^2 + d^2 \end{bmatrix} \Rightarrow tr(A^T A) = a^2 + c^2 + b^2 + d^2$$

choose $a = 1$ $d = i$, $c = b = 0$ $\Rightarrow$ $A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \neq 0$

$$\boxed{tr(A^T A) = 1 + 0 + 0 + i^2 = 1 - 1 = 0}$$

6. (15 points) Find the LU-factorization of matrix A below and use it to find the solution vector x of the linear system $\mathsf{A}\mathsf{x} = \mathsf{b}$, where

$$\mathsf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 5 & 5 \\ 6 & -3 & 5 \end{bmatrix}, \qquad \mathsf{b} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 5 & 5 \\ 6 & -3 & 5 \end{bmatrix} \xrightarrow{-2,\ -3} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & -6 & -1 \end{bmatrix} \xrightarrow{2} \boxed{\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U}, \quad \boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}}$$

$Ux = y$

$Ly = b \Rightarrow$ $\boxed{y_1 = 1}$, $2y_1 + y_2 = 4$, $3y_1 - 2y_2 + y_3 = 0$ $\Rightarrow$ $\boxed{y_2 = 2}$ $\Rightarrow$ $\boxed{y_3 = 1}$ $\quad \boxed{y = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$

$Ux = y \Rightarrow$ $\boxed{x_3 = 1}$, $3x_2 + x_3 = 2$ $\Rightarrow$ $\boxed{x_2 = \frac{1}{3}}$

$2x_1 + x_2 + 2x_3 = 1$ $\Rightarrow 2x_1 = -1 - \frac{1}{3} = -\frac{4}{3}$ $\quad \boxed{x_1 = -\frac{2}{3}}$

$$\boxed{x = \begin{bmatrix} -2/3 \\ 1/3 \\ 1 \end{bmatrix}}$$

#	Pts	Score
1	15	
2	15	
3	20	
4	20	
5	15	
6	15	
Σ	100	