Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Geometrical meaning of partial derivatives.
- The derivative of a function is a new function.
- Higher-order partial derivatives.
- The Mixed Derivative Theorem.
- Examples of implicit partial differentiation.
- Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

Next class:

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Definition

The *partial derivative with respect to x* at a point $(x, y) \in D$ of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ with values $f(x, y)$ is given by

$$f_x(x, y) = \lim_{h \to 0} \frac{1}{h} \big[f(x + h, y) - f(x, y) \big].$$

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Remark:

- To compute $f_x(x, y)$ derivate $f(x, y)$ keeping y constant.

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Remark: $f_x(x_0, y_0)$ is the slope of the line tangent to the graph of $f(x, y)$ containing the point $\big(x_0, y_0, f(x_0, y_0)\big)$ and belonging to a plane parallel to the zx-plane.

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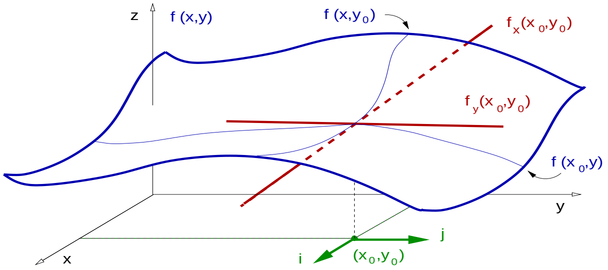

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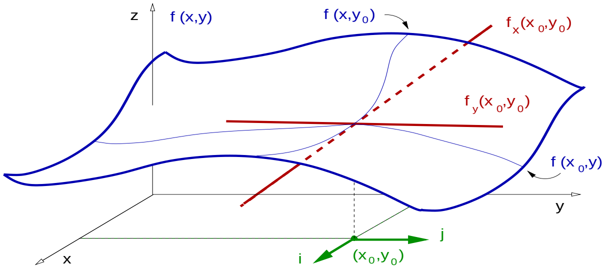

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The derivative of function $f(x) = x^2$ at an arbitrary point x is the function $f'(x) = 2x$.

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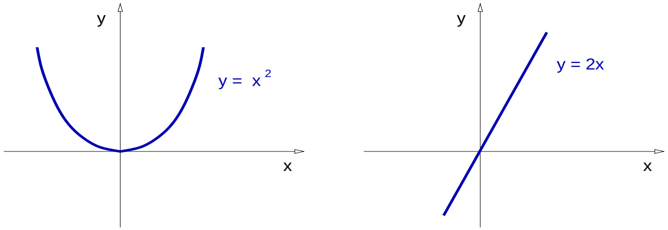

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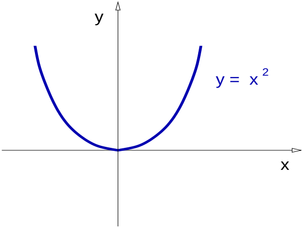

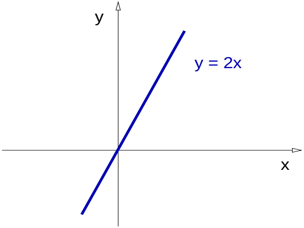

Remark: The same statement is true for partial derivatives.

The partial derivatives of a function are new functions

Definition

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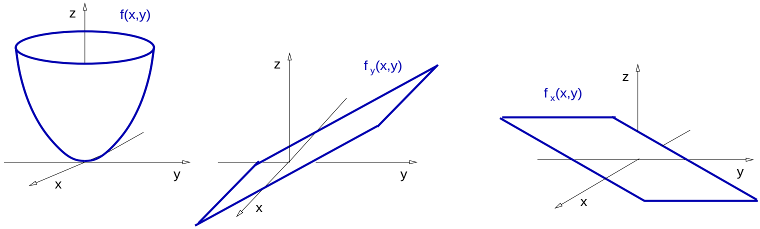

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$$f_x(x, y) = 2x, \qquad f_y(x, y) = \frac{y}{2}.$$

$\lhd$

Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Geometrical meaning of partial derivatives.
- The derivative of a function is a new function.
- **Higher-order partial derivatives.**
- The Mixed Derivative Theorem.
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If the partial derivatives f_x, f_y, f_{xy} *and* f_{yx} *of a function* $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ *exist and all are continuous functions, then holds*

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Find $\partial_x z(x, y)$ of the function z defined implicitly by the equation $xyz + e^{2z/y} + \cos(z) = 0$.

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We obtain: $(\partial_x z) = -\dfrac{yz}{\Big[xy + \frac{2}{y}\,e^{2z/y} - \sin(z)\Big]}$. $\triangleleft$

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Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$

Definition

The *partial derivative with respect to x_i* at a point $(x_1, \cdots, x_n) \in D$ of a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$ and $i = 1, \cdots, n$, is given by

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Notation: $f_{x_i}, \qquad f_i, \qquad \dfrac{\partial f}{\partial x_i}, \qquad \partial_{x_i} f, \qquad \partial_i f.$

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Compute all first partial derivatives of the function
$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$.

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Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$

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Verify that $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation: $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$.

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Analogously, $\phi_{yy} = -\dfrac{1}{r^3} + \dfrac{3y^2}{r^5}$, and $\phi_{zz} = -\dfrac{1}{r^3} + \dfrac{3z^2}{r^5}$.

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Denote $r = \sqrt{x^2 + y^2 + z^2}$, then $\phi_{xx} = -\dfrac{1}{r^3} + \dfrac{3x^2}{r^5}$.

Analogously, $\phi_{yy} = -\dfrac{1}{r^3} + \dfrac{3y^2}{r^5}$, and $\phi_{zz} = -\dfrac{1}{r^3} + \dfrac{3z^2}{r^5}$. Then,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5}$$

Partial derivatives of $f : D \subset \mathbb{R}^n \to \mathbb{R}$

Example

Verify that $\phi(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ satisfies the Laplace equation: $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$

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We conclude that $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$. $\triangleleft$

Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- Differentiability and continuity.
- A primer on differential equations.

Partial derivatives and continuity

Recall: The following result holds for single variable functions.

Theorem

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Remark: However, the claim "If $f_x(x,y)$ and $f_y(x,y)$ exist, then $f(x,y)$ is continuous" is false.

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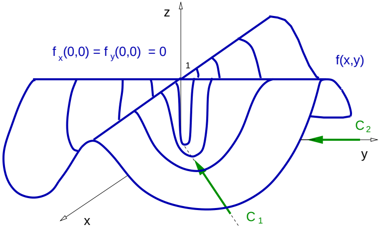

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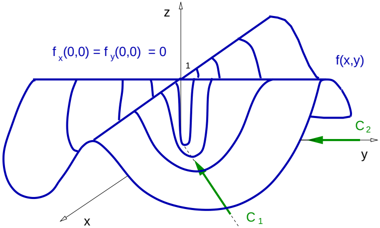

Remark: This is a bad property for a differentiable function.

Partial derivatives and continuity

Remark: Here is another discontinuous function at $(0, 0)$ having partial derivatives at $(0, 0)$.

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Example

(a) Show that f is not continuous at $(0,0)$;

(b) Find $f_x(0,0)$ and $f_y(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

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The Two-Path Theorem implies that $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

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Therefore, $f_x(0,0) = f_y(0,0) = 0$. $\triangleleft$

Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives and continuity.
- **Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.**
- Differentiability and continuity.
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Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Recall: A differentiable function $f : \mathbb{R} \to \mathbb{R}$ at x_0 must be approximated by a line $L(x)$ by $(x_0, f(x_0))$ with slope $f'(x_0)$.

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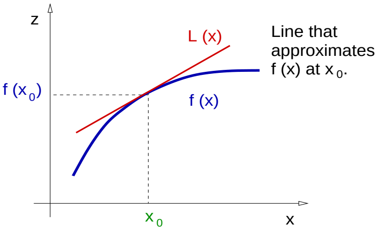

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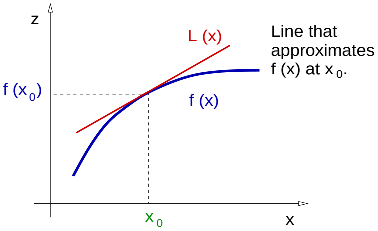

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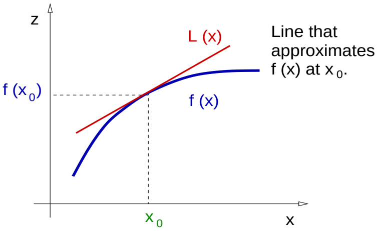

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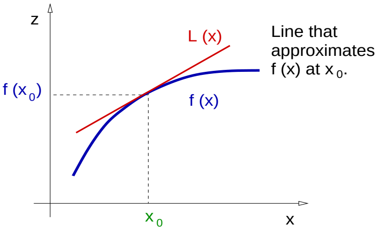

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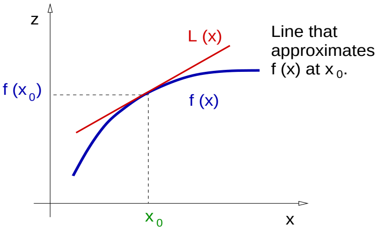

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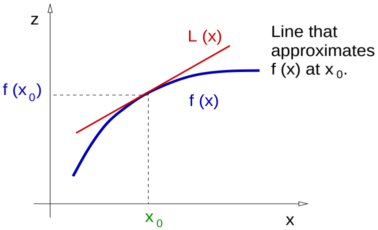

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Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Remark: The idea to define differentiable functions:
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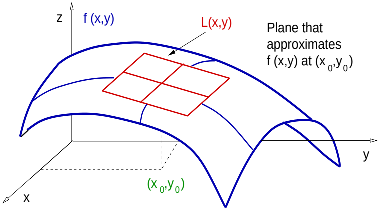

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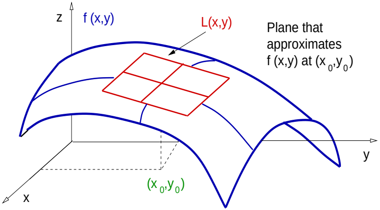

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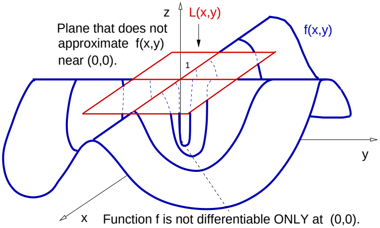

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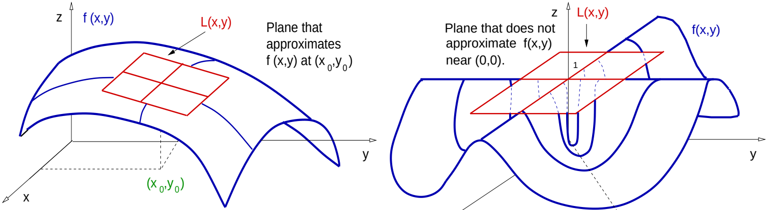

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We will show next week that the equation of the plane L is

$$L(x,y) = f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0) + f(x_0,y_0).$$

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Definition

Given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and an interior point (x_0, y_0) in D, let L be the linear function

$$L(x, y) = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0) + f(x_0, y_0).$$

The function f is called *differentiable at* (x_0, y_0) iff the function f is approximated by the linear function L near (x_0, y_0), that is,

$$f(x, y) = L(x, y) + \epsilon_1\,(x - x_0) + \epsilon_2\,(y - y_0)$$

where the functions ϵ_1 and $\epsilon_2 \to 0$ as $(x, y) \to (x_0, y_0)$.
The function f is *differentiable* iff f is differentiable at every interior point of D.

Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

Remark: Recalling the linear function L given above,

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$$\Delta z = (z - z_0), \quad \Delta y = (y - y_0), \quad \Delta x = (x - x_0);$$

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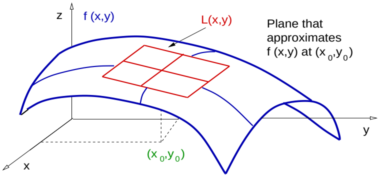
(Equation used in the textbook to define a differentiable function.)

Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
- **Differentiability and continuity.**
- A primer on differential equations.

Differentiability and continuity

Remark: We will show in Sect. 14.6 that the graph of a differentiable function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is approximated by a plane at every point in D.



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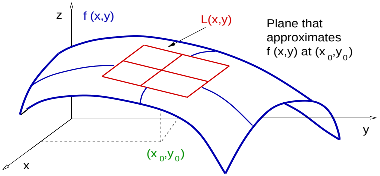

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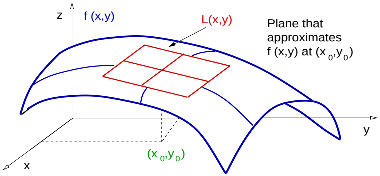

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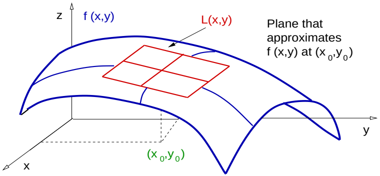

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Theorem

If the partial derivatives f_x and f_y of a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ are continuous in an open region $R \subset D$, then f is differentiable in R.

Partial derivatives and differentiability (Sect. 14.3)

- Partial derivatives and continuity.
- Differentiable functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
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$$f'(x) = k\, f(x).$$

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Therefore, $f(x) = c\, e^{kx}$. $\triangleleft$

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- The Wave equation: (Light, sound, gravitation.)

$$f_{tt} = v\left(f_{xx} + f_{yy} + f_{zz}\right).$$

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We conclude that $T_t = T_{xx}$. $\triangleleft$

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We conclude that $f_{xx} + f_{yy} + f_{zz} = 0$. ◁

A primer on differential equations

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Since $v^2 f_{xx} = 6v^2\,(vt - x)$, then $f_{tt} = v^2 f_{xx}$. $\lhd$

A primer on differential equations

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Given any $v \in \mathbb{R}$ and any twice continuously differentiable function $u : \mathbb{R} \to \mathbb{R}$, verify that $f(t, x) = u(vt - x)$, satisfies the one-space dimensional wave equation $f_{tt} = v^2 f_{xx}$.

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Since $v^2 f_{xx} = v^2 u''(vt - x)$, then $f_{tt} = v^2 f_{xx}$. $\triangleleft$

Chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: Chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - Chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - Chain rule for functions defined on a curve in a plane.
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- A formula for implicit differentiation.

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Theorem

If the functions $f : [x_0, x_1] \to \mathbb{R}$ *and* $x : [t_0, t_1] \to [x_0, x_1]$ *are differentiable, then the function* $\hat{f} : [t_0, t_1] \to \mathbb{R}$ *given by the composition* $\hat{f}(t) = f\big(x(t)\big)$ *is differentiable and*

$$\frac{d\hat{f}}{dt}(t) = \frac{df}{dx}\big(x(t)\big)\,\frac{dx}{dt}(t).$$

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Review: The chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$

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Example

The volume V of a gas balloon depends on the temperature F in Fahrenheit as $V(F) = k\,F^2 + V_0$. Let $F(C) = (9/5)C + 32$ be the temperature in Fahrenheit corresponding to C in Celsius. Find the rate of change $\hat{V}'(C)$.

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We conclude that $V'(C) = \frac{18k}{5}\left(\frac{9}{5}C + 32\right)$. $\triangleleft$

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Chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: Chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
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- **Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.**
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Theorem

If the functions $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ *and* $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^2$ *are differentiable, with* $\mathbf{r}(t) = \langle x(t), y(t) \rangle$*, then the function* $\hat{f} : \mathbb{R} \to \mathbb{R}$ *given by the composition* $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ *is differentiable and holds*

$$\frac{d\hat{f}}{dt}(t) = \frac{\partial f}{\partial x}\big(\mathbf{r}(t)\big)\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}\big(\mathbf{r}(t)\big)\,\frac{dy}{dt}(t).$$

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An alternative notation is $\hat{f}' = f_x\,x' + f_y\,y'$.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

The chain rule for functions defined on a curve in a plane.

Example

Find the rate of change of the function $f(x, y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \sin(t), \cos(2t) \rangle$.

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Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.

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Since $x(t) = \sin(t)$ and $y(t) = \cos(2t)$,

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Since $x(t) = \sin(t)$ and $y(t) = \cos(2t)$,

$$\hat{f}'(t) = 2\sin(t)\, \cos(t) + 6\cos^2(2t)\, \big[-2\sin(2t)\big].$$

The chain rule for functions defined on a curve in a plane.

Example

Find the rate of change of the function $f(x, y) = x^2 + 2y^3$, along the curve $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle \sin(t), \cos(2t) \rangle$.

Solution: The rate of change of f along the curve $\mathbf{r}(t)$ is the derivative of $\hat{f}(t) = f(\mathbf{r}(t)) = f(x(t), y(t))$. We do not need to compute $\hat{f}(t) = f(\mathbf{r}(t))$. Instead, the chain rule implies

$$\hat{f}'(t) = f_x(\mathbf{r})\, x' + f_y(\mathbf{r})\, y' = 2x\, x' + 6y^2\, y'.$$

Since $x(t) = \sin(t)$ and $y(t) = \cos(2t)$,

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The result is $\hat{f}'(t) = 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t)$. $\triangleleft$

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

The chain rule for change of coordinates in a plane.

Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$

The chain rule for change of coordinates in a plane.

Theorem

If the functions $f : \mathbb{R}^2 \to \mathbb{R}$ *and the change of coordinate functions* $x, y : \mathbb{R}^2 \to \mathbb{R}$ *are differentiable, with* $x(t,s)$ *and* $y(t,s)$*, then the function* $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ *given by the composition* $\hat{f}(t,s) = f\big(x(t,s), y(t,s)\big)$ *is differentiable and holds*

$$\hat{f}_t = f_x\, x_t + f_y\, y_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s.$$

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$$\hat{f}_t = f_x\, x_t + f_y\, y_t$$

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Remark: We denote by $f(x,y)$ the function values in the coordinates (x,y), while we denote by $\hat{f}(t,s)$ are the function values in the coordinates (t,s).

The chain rule for change of coordinates in a plane.

Example

Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find the derivatives of f in polar coordinates (r, θ).

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The chain rule for change of coordinates in a plane.

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Given the function $f(x, y) = x^2 + 3y^2$, in Cartesian coordinates (x, y), find the derivatives of f in polar coordinates (r, θ).

Solution: The relation between Cartesian and polar coordinates is

$$x(r, \theta) = r\cos(\theta), \quad y(r, \theta) = r\sin(\theta).$$

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$$\hat{f}_r$$

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$$\hat{f}_\theta = -2r^2\cos(\theta)\sin(\theta) + 6r^2\cos(\theta)\sin(\theta). \qquad \lhd$$

Chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: Chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - Chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - Chain rule for functions defined on a curve in a plane.
  - Chain rule for change of coordinates in a plane.
- **Functions of three variables,** $f : D \subset \mathbb{R}^3 \to \mathbb{R}$**.**
  - Chain rule for functions defined on a curve in space.
  - Chain rule for functions defined on surfaces in space.
  - Chain rule for change of coordinates in space.
- A formula for implicit differentiation.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

Chain rule for functions defined on a curve in space.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.

Chain rule for functions defined on a curve in space.

Theorem

If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ *are differentiable, with* $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$*, then the function* $\hat{f} : \mathbb{R} \to \mathbb{R}$ *given by the composition* $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ *is differentiable and holds*

$$\frac{d\hat{f}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

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If the functions $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to D \subset \mathbb{R}^3$ are differentiable, with $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ given by the composition $\hat{f}(t) = f\big(\mathbf{r}(t)\big)$ is differentiable and holds

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Notation:

The equation above is usually written as

$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on a curve in space.

Example

Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

Solution: Recall: We do not need to compute

$$\hat{f}(t) = f(\mathbf{r}(t))$$

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Find the derivative of $f = x^2 + y^3 + z^4$ along the curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$.

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to find $\hat{f}'$.

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$$\hat{f}' = -2x\sin(t) +$$

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$$\hat{f}' = f_x\, x' + f_y\, y' + f_z\, z'.$$

$$\hat{f}' = -2x \sin(t) + 3y^2 \cos(t) + 4z^3(3).$$

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$$\hat{f}' = -2x \sin(t) + 3y^2 \cos(t) + 4z^3(3).$$

$$\hat{f}' = -2\cos(t)\sin(t) + 3\sin^2(t)\cos(t) + 4(3)(3^3)t^3. \qquad \triangleleft$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ *and the surface given by functions* $x, y, z : \mathbb{R}^2 \to \mathbb{R}$ *are differentiable, with* $x(t,s)$ *and* $y(t,s)$*, and* $z(t,s)$*, then the function* $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ *given by the composition* $\hat{f}(t,s) = f\big(x(t,s), y(t,s), z(t,s)\big)$ *is differentiable and holds*

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$
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$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t,$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s.$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s)$ the function values at the surface point with coordinates (t,s).

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find its derivatives on the surface given by $x(t, s) = t + s, \quad y(t, s) = t^2 + s^2, \quad z(t, s) = t - s.$

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Solution: Recall: We do not need to compute the function

$$\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$$

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Solution: Recall: We do not need to compute the function

$$\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$$

to obtain the derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$,

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Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find its derivatives on the surface given by $x(t, s) = t + s, \quad y(t, s) = t^2 + s^2, \quad z(t, s) = t - s.$

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$$\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$$

to obtain the derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$, which are given by $\hat{f}_t$ and $\hat{f}_s$. We just use the chain rule,

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for functions defined on surfaces in space.

Example

Given the function $f(x, y) = x^2 + 3y^2 + 2z^2$, in Cartesian coordinates (x, y), find its derivatives on the surface given by $x(t, s) = t + s, \quad y(t, s) = t^2 + s^2, \quad z(t, s) = t - s.$

Solution: Recall: We do not need to compute the function

$$\hat{f}(t, s) = f(x(t, s), y(t, s), z(t, s))$$

to obtain the derivatives of f along the surface $x(t, s)$, $y(t, s)$ and $z(t, s)$, which are given by $\hat{f}_t$ and $\hat{f}_s$. We just use the chain rule,

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Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

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$\hat{f}_t$

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Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Remark: We describe the surface in the previous example.

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Given the surface in parametric form by the equations

$$x(t,s) = t + s, \qquad y(t,s) = t^2 + s^2, \qquad z(t,s) = t - s,$$

express that surface as an equation for x, y and z.

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We introduce these t and s into the equation for y,

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hence, $y = \dfrac{x^2}{2} + \dfrac{z^2}{2}$, a circular paraboloid along the y axis. $\triangleleft$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for change of coordinates in space.

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

Chain rule for change of coordinates in space.

Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ *and the change of coordinate functions* $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ *are differentiable, with* $x(t, s, r)$*,* $y(t, s, r)$*, and* $z(t, s, r)$*, then the function* $\hat{f} : \mathbb{R}^3 \to \mathbb{R}$ *given by the composition* $\hat{f}(t, s, r) = f\big(x(t, s, r), y(t, s, r), z(t, s, r)\big)$ *is differentiable and*

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t$$
$$\hat{f}_s = f_x\, x_s + f_y\, y_s + f_z\, z_s$$
$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r.$$

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Theorem

If the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and the change of coordinate functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ are differentiable, with $x(t,s,r)$, $y(t,s,r)$, and $z(t,s,r)$, then the function $\hat{f} : \mathbb{R}^3 \to \mathbb{R}$ given by the composition $\hat{f}(t,s,r) = f\big(x(t,s,r), y(t,s,r), z(t,s,r)\big)$ is differentiable and

$$\hat{f}_t = f_x\, x_t + f_y\, y_t + f_z\, z_t$$
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$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r.$$

Remark:

We denote by $f(x,y,z)$ the function values in the coordinates (x,y,z), while we denote by $\hat{f}(t,s,r)$ the function values in the coordinates (t,s,r).

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Given the function $f(x, y, z) = x^2 + 3y^2 + z^2$, in Cartesian coordinates, find its r-derivative in spherical coordinates (r, θ, ϕ),

$$x = r\cos(\phi)\sin(\theta), \quad y = r\sin(\phi)\sin(\theta), \quad z = r\cos(\theta).$$

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Solution: Recall: We do not need to compute the function

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to obtain the r-derivative of f.

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$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r = 2x\, x_r + 6y\, y_r + 2z\, z_r$$

Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$

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$$\hat{f}_r = f_x\, x_r + f_y\, y_r + f_z\, z_r = 2x\, x_r + 6y\, y_r + 2z\, z_r$$

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$$\hat{f}_r = 2r\sin^2(\theta) + 4r\sin^2(\phi)\sin^2(\theta) + 2r\cos^2(\theta).$$

We conclude that $\hat{f}_r = 2r + 4r\sin^2(\phi)\sin^2(\theta)$. $\triangleleft$

Chain rule for functions of 2, 3 variables (Sect. 14.4)

- Review: Chain rule for $f : D \subset \mathbb{R} \to \mathbb{R}$.
  - Chain rule for change of coordinates in a line.
- Functions of two variables, $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  - Chain rule for functions defined on a curve in a plane.
  - Chain rule for change of coordinates in a plane.
- Functions of three variables, $f : D \subset \mathbb{R}^3 \to \mathbb{R}$.
  - Chain rule for functions defined on a curve in space.
  - Chain rule for functions defined on surfaces in space.
  - Chain rule for change of coordinates in space.
- **A formula for implicit differentiation.**

A formula for implicit differentiation

Theorem

If the differentiable function with values $F(x, y)$ defines implicitly the function values $y(x)$ by the equation $F(x, y) = 0$, and if the function $F_y \neq 0$, then y is differentiable and

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

A formula for implicit differentiation

Theorem

If the differentiable function with values $F(x, y)$ defines implicitly the function values $y(x)$ by the equation $F(x, y) = 0$, and if the function $F_y \neq 0$, then y is differentiable and

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Proof: Since $y(x)$ are defined implicitly by the equation

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Example

Find the derivative of function $y : \mathbb{R} \to \mathbb{R}$ defined implicitly by the equation $F(x, y) = 0$, where $F(x, y) = x\, e^{y} + \cos(x - y)$.

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Therefore, the Theorem above implies

$$y'(x) = \frac{\big[\sin(x - y) - e^y\big]}{\big[x\,e^y + \sin(x - y)\big]}.$$

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We conclude that: $y'(x) = \dfrac{\left[\sin(x - y) - e^y\right]}{\left[x\, e^y + \sin(x - y)\right]}.$ $\triangleleft$