Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- Scalar and vector projection formulas.

Two main ways to introduce the dot product

Geometrical
definition

Two main ways to introduce the dot product

Geometrical definition $\longrightarrow$ Properties

Two main ways to introduce the dot product

Geometrical definition $\longrightarrow$ Properties $\longrightarrow$ Expression in components.

Two main ways to introduce the dot product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.

Definition in components

Two main ways to introduce the dot product

Geometrical definition $\longrightarrow$ Properties $\longrightarrow$ Expression in components.

Definition in components $\longrightarrow$ Properties

Two main ways to introduce the dot product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.

Definition in components	$\longrightarrow$	Properties	$\longrightarrow$	Geometrical expression.

Two main ways to introduce the dot product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.

Definition in components	$\longrightarrow$	Properties	$\longrightarrow$	Geometrical expression.

We choose the first way, the textbook chooses the second way.

Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- **Geometric definition of dot product.**
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- Scalar and vector projection formulas.

The dot product of two vectors is a scalar

Definition

The *dot product* of the vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, with $n = 2, 3$, having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \cdot \mathbf{w}$ and given by

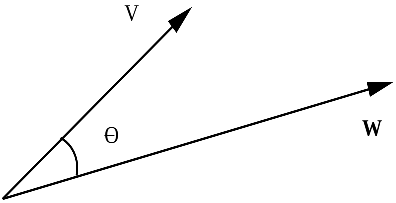
$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \, |\mathbf{w}| \cos(\theta).$$

The dot product of two vectors is a scalar

Definition

The *dot product* of the vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, with $n = 2, 3$, having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \cdot \mathbf{w}$ and given by

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\, |\mathbf{w}| \cos(\theta).$$



Initial points together.

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

$$|\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \quad \Rightarrow \quad |\mathbf{w}| = \sqrt{14}.$$

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

$$|\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \quad \Rightarrow \quad |\mathbf{w}| = \sqrt{14}.$$

We now use the definition of dot product:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = (2)\,\sqrt{14}\,\frac{\sqrt{2}}{2} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = 2\sqrt{7}.$$

$\lhd$

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

$$|\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \quad \Rightarrow \quad |\mathbf{w}| = \sqrt{14}.$$

We now use the definition of dot product:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = (2)\,\sqrt{14}\,\frac{\sqrt{2}}{2} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = 2\sqrt{7}.$$

$\triangleleft$

- The angle between two vectors usually is not know in applications.

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

$$|\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \quad \Rightarrow \quad |\mathbf{w}| = \sqrt{14}.$$

We now use the definition of dot product:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = (2)\,\sqrt{14}\,\frac{\sqrt{2}}{2} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = 2\sqrt{7}.$$

$\triangleleft$

- The angle between two vectors usually is not know in applications.
- It is useful to have a formula for the dot product involving the vector components.

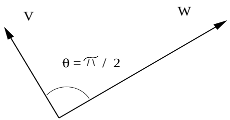
Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- **Orthogonal vectors.**
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- Scalar and vector projection formulas.

Perpendicular vectors have zero dot product.

Definition

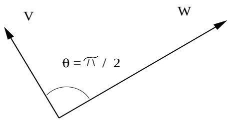
Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



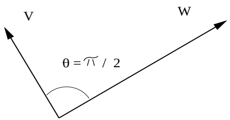
Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

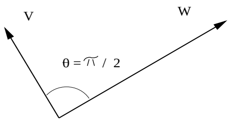
Proof.

$$0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$$

Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

Proof.

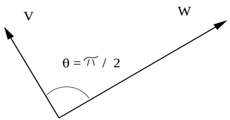
$$0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$$

$$|\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0$$

Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

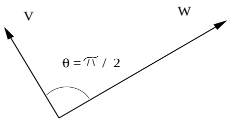
Proof.

$$\left.\begin{array}{c} 0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \\ |\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0 \end{array}\right\} \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \cos(\theta) = 0 \\ \\ \end{array}\right.$$

Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

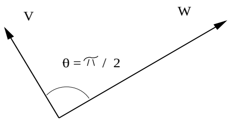
Proof.

$$\left.\begin{array}{c} 0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \\ |\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} \cos(\theta) = 0 \\ 0 \leqslant \theta \leqslant \pi \end{array}\right.$$

Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

Proof.

$$\left.\begin{array}{c} 0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \\ |\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} \cos(\theta) = 0 \\ 0 \leqslant \theta \leqslant \pi \end{array}\right. \quad \Leftrightarrow \quad \theta = \frac{\pi}{2}.$$

$\square$

The dot product of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is simple to compute

Example

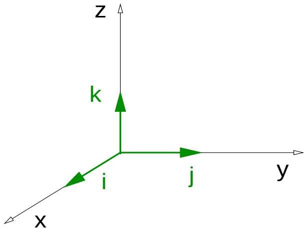
Compute all dot products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

The dot product of **i**, **j** and **k** is simple to compute

Example

Compute all dot products involving the vectors **i**, **j** , and **k**.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

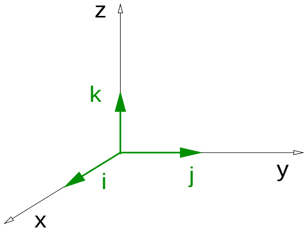


The dot product of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is simple to compute

Example

Compute all dot products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



$$\mathbf{i} \cdot \mathbf{i} = 1, \qquad \mathbf{j} \cdot \mathbf{j} = 1, \qquad \mathbf{k} \cdot \mathbf{k} = 1,$$
$$\mathbf{i} \cdot \mathbf{j} = 0, \qquad \mathbf{j} \cdot \mathbf{i} = 0, \qquad \mathbf{k} \cdot \mathbf{i} = 0,$$
$$\mathbf{i} \cdot \mathbf{k} = 0, \qquad \mathbf{j} \cdot \mathbf{k} = 0, \qquad \mathbf{k} \cdot \mathbf{j} = 0.$$

◁

Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- **Dot product and orthogonal projections.**
- Properties of the dot product.
- Dot product in vector components.
- Scalar and vector projection formulas.

The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

The dot product and orthogonal projections.

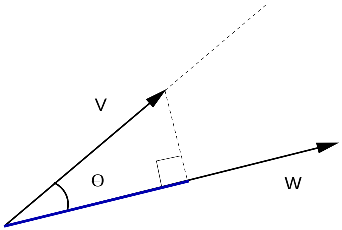
Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.

The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

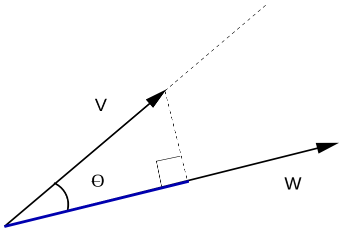
Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.



The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\, |\mathbf{w}| \cos(\theta)$.

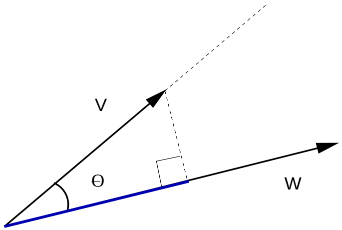


$|\mathbf{v}| \cos(\theta)$

The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.

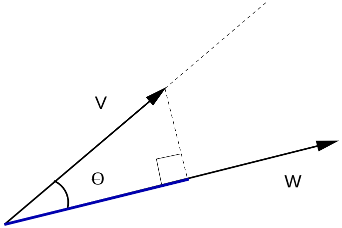


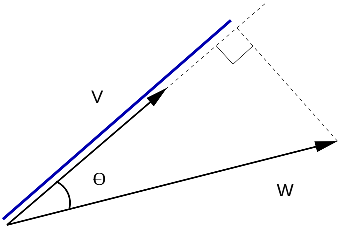
$$|\mathbf{v}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.



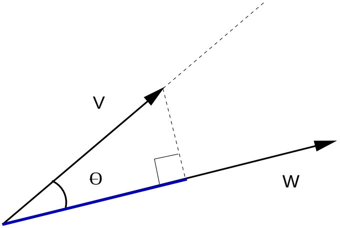


$$|\mathbf{v}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

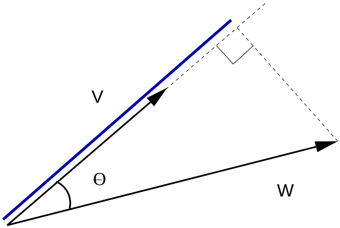
The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.



$$|\mathbf{v}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

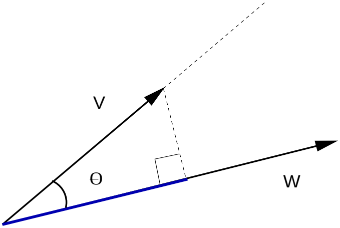


$$|\mathbf{w}|\,\cos(\theta)$$

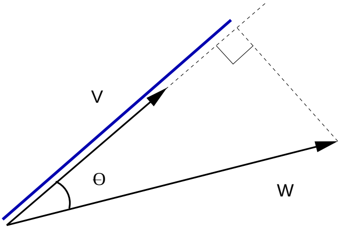
The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.



$$|\mathbf{v}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

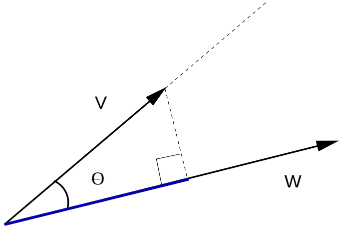


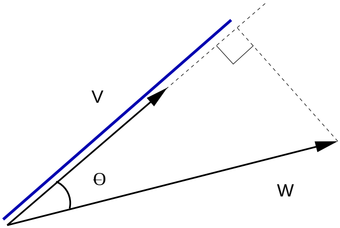
$$|\mathbf{w}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|}.$$

The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.





$$|\mathbf{v}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

$$|\mathbf{w}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|}.$$

Remark: If $|\mathbf{u}| = 1$, then $\mathbf{v} \cdot \mathbf{u}$ is the projection of $\mathbf{v}$ along $\mathbf{u}$.

Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- **Properties of the dot product.**
- Dot product in vector components.
- Scalar and vector projection formulas.

Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric)*;

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear)*;

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear)*;

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive)*;

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric)*;

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear)*;

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear)*;

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive)*;

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

Proof.

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.

Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric)*;

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear)*;

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear)*;

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive)*;

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

Proof.

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.
For example, the proof of (b) for $a > 0$:

Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric)*;

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear)*;

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear)*;

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive)*;

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

Proof.

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.
For example, the proof of (b) for $a > 0$:

$$\mathbf{v} \cdot (a\mathbf{w}) = |\mathbf{v}|\,|a\mathbf{w}|\,\cos(\theta)$$

Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric)*;

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear)*;

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear)*;

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive)*;

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

Proof.

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.
For example, the proof of (b) for $a > 0$:

$$\mathbf{v} \cdot (a\mathbf{w}) = |\mathbf{v}|\,|a\mathbf{w}|\,\cos(\theta) = a\,|\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$$

Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric)*;

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear)*;

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear)*;

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive)*;

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

Proof.

Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.
For example, the proof of (b) for $a > 0$:

$$\mathbf{v} \cdot (a\mathbf{w}) = |\mathbf{v}|\,|a\mathbf{w}|\,\cos(\theta) = a\,|\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = a\,(\mathbf{v} \cdot \mathbf{w}).$$

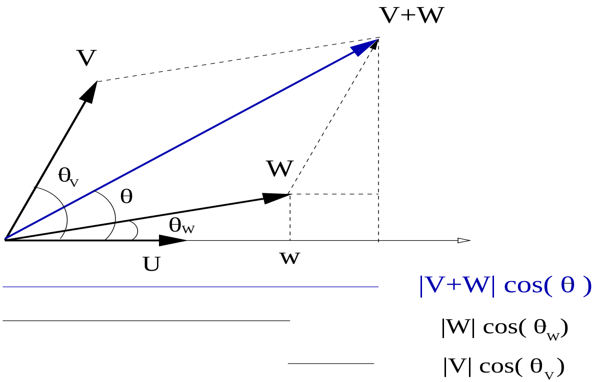
$\square$

Properties of the dot product.

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, is non-trivial.

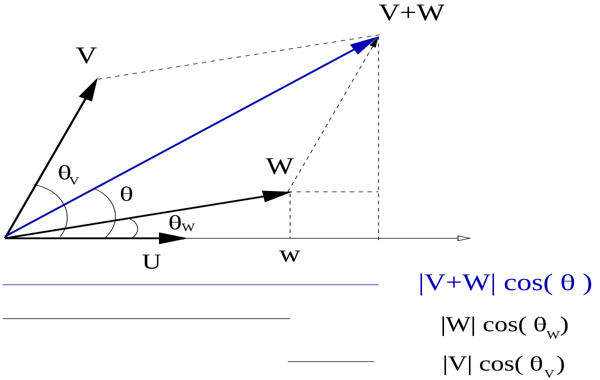
Properties of the dot product.

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, is non-trivial. The proof is:



Properties of the dot product.

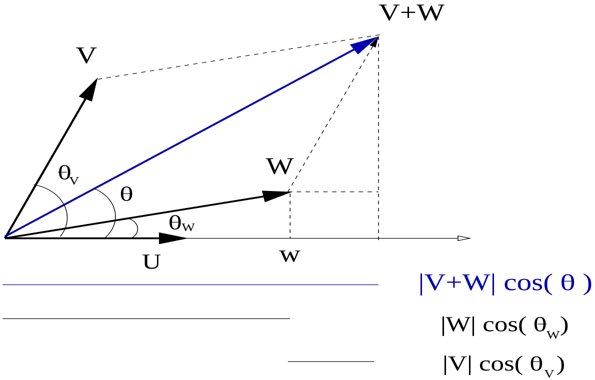
(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, is non-trivial. The proof is:



$$|\mathbf{v} + \mathbf{w}| \cos(\theta) = \frac{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})}{|\mathbf{u}|},$$

Properties of the dot product.

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, is non-trivial. The proof is:

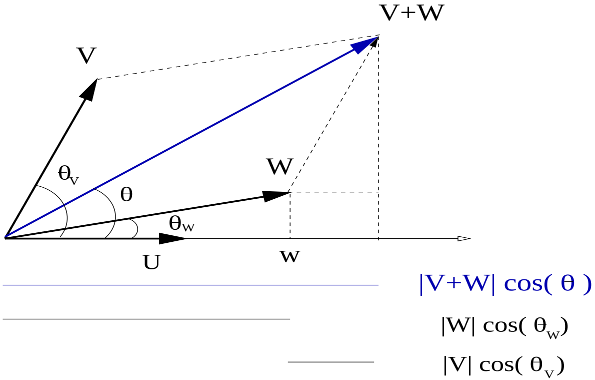


$$|\mathbf{v} + \mathbf{w}| \cos(\theta) = \frac{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})}{|\mathbf{u}|},$$

$$|\mathbf{w}| \cos(\theta_w) = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}|},$$

Properties of the dot product.

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, is non-trivial. The proof is:



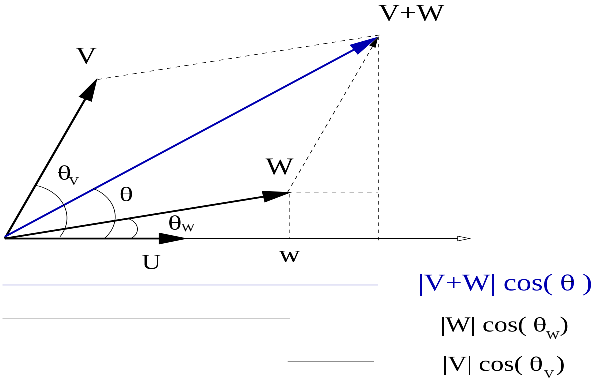
$$|\mathbf{v} + \mathbf{w}| \cos(\theta) = \frac{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})}{|\mathbf{u}|},$$

$$|\mathbf{w}| \cos(\theta_w) = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}|},$$

$$|\mathbf{v}| \cos(\theta_v) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|},$$

Properties of the dot product.

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, is non-trivial. The proof is:



$$\left.\begin{aligned} |\mathbf{v}+\mathbf{w}|\cos(\theta) &= \frac{\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})}{|\mathbf{u}|},\\ |\mathbf{w}|\cos(\theta_w) &= \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{u}|},\\ |\mathbf{v}|\cos(\theta_v) &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}, \end{aligned}\right\} \Rightarrow \mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$$

Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- **Dot product in vector components.**
- Scalar and vector projection formulas.

The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y .$$

The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

Proof.

Recall: $\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j}$ and $\mathbf{w} = w_x\,\mathbf{i} + w_y\,\mathbf{j}$.

The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

Proof.

Recall: $\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j}$ and $\mathbf{w} = w_x\,\mathbf{i} + w_y\,\mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x\,\mathbf{i} + v_y\,\mathbf{j}) \cdot (w_x\,\mathbf{i} + w_y\,\mathbf{j})$$

The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y .$$

Proof.

Recall: $\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j}$ and $\mathbf{w} = w_x\,\mathbf{i} + w_y\,\mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x\,\mathbf{i} + v_y\,\mathbf{j}\,) \cdot (w_x\,\mathbf{i} + w_y\,\mathbf{j}\,)$$

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x\,\mathbf{i} \cdot \mathbf{i} + v_x w_y\,\mathbf{i} \cdot \mathbf{j} + v_y w_x\,\mathbf{j} \cdot \mathbf{i} + v_y w_y\,\mathbf{j} \cdot \mathbf{j} .$$

The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

Proof.

Recall: $\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j}$ and $\mathbf{w} = w_x\,\mathbf{i} + w_y\,\mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x\,\mathbf{i} + v_y\,\mathbf{j}) \cdot (w_x\,\mathbf{i} + w_y\,\mathbf{j})$$

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x\,\mathbf{i} \cdot \mathbf{i} + v_x w_y\,\mathbf{i} \cdot \mathbf{j} + v_y w_x\,\mathbf{j} \cdot \mathbf{i} + v_y w_y\,\mathbf{j} \cdot \mathbf{j}.$$

Recall: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$.

The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

Proof.

Recall: $\mathbf{v} = v_x \,\mathbf{i} + v_y \,\mathbf{j}$ and $\mathbf{w} = w_x \,\mathbf{i} + w_y \,\mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x \,\mathbf{i} + v_y \,\mathbf{j}) \cdot (w_x \,\mathbf{i} + w_y \,\mathbf{j})$$

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x \,\mathbf{i} \cdot \mathbf{i} + v_x w_y \,\mathbf{i} \cdot \mathbf{j} + v_y w_x \,\mathbf{j} \cdot \mathbf{i} + v_y w_y \,\mathbf{j} \cdot \mathbf{j}.$$

Recall: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$. We conclude that

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

□

The dot product in vector components (Case $\mathbb{R}^3$)

Theorem

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *and* $\mathbf{w} = \langle w_x, w_y, w_z \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z .$$

The dot product in vector components (Case $\mathbb{R}^3$)

Theorem

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *and* $\mathbf{w} = \langle w_x, w_y, w_z \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

- The proof is similar to the case in $\mathbb{R}^2$.

The dot product in vector components (Case $\mathbb{R}^3$)

Theorem

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *and* $\mathbf{w} = \langle w_x, w_y, w_z \rangle$*, then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

- The proof is similar to the case in $\mathbb{R}^2$.
- The dot product is simple to compute from the vector component formula $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$.

The dot product in vector components (Case $\mathbb{R}^3$)

Theorem

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *and* $\mathbf{w} = \langle w_x, w_y, w_z \rangle$*, then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z .$$

- The proof is similar to the case in $\mathbb{R}^2$.
- The dot product is simple to compute from the vector component formula $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$.
- The geometrical meaning of the dot product is simple to see from the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$$

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Furthermore,

$$\mathbf{v} \cdot \mathbf{w} = (1)(2) + (2)(1)$$

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Furthermore,

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (1)(2) + (2)(1) \\ |\mathbf{v}| &= \sqrt{1^2 + 2^2} = \sqrt{5}, \end{aligned}$$

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Furthermore,

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (1)(2) + (2)(1) \\ |\mathbf{v}| &= \sqrt{1^2 + 2^2} = \sqrt{5}, \\ |\mathbf{w}| &= \sqrt{2^2 + 1^2} = \sqrt{5}, \end{aligned}$$

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Furthermore,

$$\left.\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (1)(2) + (2)(1) \\ |\mathbf{v}| &= \sqrt{1^2 + 2^2} = \sqrt{5}, \\ |\mathbf{w}| &= \sqrt{2^2 + 1^2} = \sqrt{5}, \end{aligned}\right\} \quad \Rightarrow \quad \cos(\theta) = \frac{4}{5}.$$

$\lhd$

Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- **Scalar and vector projection formulas.**

Scalar and vector projection formulas.

Theorem

The scalar projection of $\mathbf{v}$ *along* $\mathbf{w}$ *is the number* $p_w(v)$,

$$p_w(v) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

The vector projection of $\mathbf{v}$ *along* $\mathbf{w}$ *is the vector* $\mathbf{p}_w(v)$,

$$\mathbf{p}_w(v) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}\right) \frac{\mathbf{w}}{|\mathbf{w}|}.$$

Scalar and vector projection formulas.

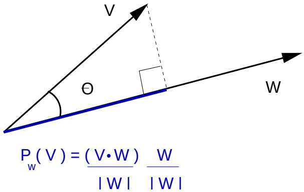
Theorem

The scalar projection of $\mathbf{v}$ *along* $\mathbf{w}$ *is the number* $p_w(v)$,

$$p_w(v) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

The vector projection of $\mathbf{v}$ *along* $\mathbf{w}$ *is the vector* $\mathbf{p}_w(v)$,

$$\mathbf{p}_w(v) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}\right) \frac{\mathbf{w}}{|\mathbf{w}|}.$$



Scalar and vector projection formulas.

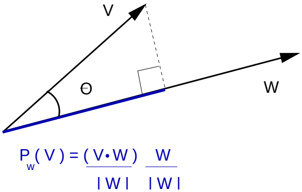
Theorem

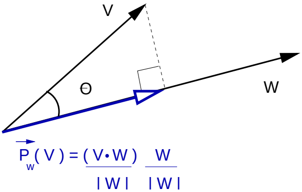
The scalar projection of $\mathbf{v}$ *along* $\mathbf{w}$ *is the number* $p_w(v)$,

$$p_w(v) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

The vector projection of $\mathbf{v}$ *along* $\mathbf{w}$ *is the vector* $\mathbf{p}_w(v)$,

$$\mathbf{p}_w(v) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}\right) \frac{\mathbf{w}}{|\mathbf{w}|}.$$





Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta)$$

Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}$$

Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The scalar projection of **b** onto **a** is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$

Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$

We therefore obtain $p_a(b) = -\frac{2}{\sqrt{5}}$.

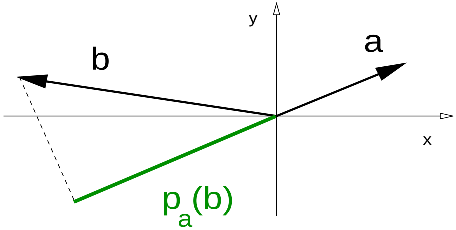
Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1\rangle$ onto $\mathbf{a} = \langle 1, 2\rangle$.

Solution: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$

We therefore obtain $p_a(b) = -\frac{2}{\sqrt{5}}$.



Example

Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Example

Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$\mathbf{p}_a(b) = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example

Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$\mathbf{p}_a(b) = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(-\frac{2}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} \langle 1, 2 \rangle.$$

Example

Find the vector projection of $\mathbf{b} = \langle -4, 1\rangle$ onto $\mathbf{a} = \langle 1, 2\rangle$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$\mathbf{p}_a(b) = \left(\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|}\right)\,\frac{\mathbf{a}}{|\mathbf{a}|} = \Big(-\frac{2}{\sqrt{5}}\Big)\,\frac{1}{\sqrt{5}}\,\langle 1, 2\rangle.$$

We therefore obtain $\mathbf{p}_a(b) = -\left\langle \frac{2}{5}, \frac{4}{5}\right\rangle$.

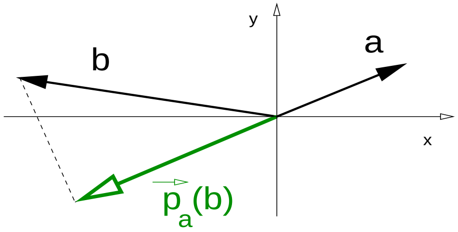
Example

Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The vector projection of **b** onto **a** is the vector

$$\mathbf{p}_a(b) = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(-\frac{2}{\sqrt{5}} \right) \frac{1}{\sqrt{5}} \langle 1, 2 \rangle.$$

We therefore obtain $\mathbf{p}_a(b) = -\left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$.



Example

Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Example

Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Solution: The vector projection of **a** onto **b** is the vector

$$\mathbf{p}_b(a) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \frac{\mathbf{b}}{|\mathbf{b}|}$$

Example

Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Solution: The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is the vector

$$\mathbf{p}_b(a) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\right) \frac{\mathbf{b}}{|\mathbf{b}|} = \left(-\frac{2}{\sqrt{17}}\right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle.$$

Example

Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Solution: The vector projection of **a** onto **b** is the vector

$$\mathbf{p}_b(a) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \frac{\mathbf{b}}{|\mathbf{b}|} = \left(-\frac{2}{\sqrt{17}} \right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle.$$

We therefore obtain $\mathbf{p}_a(b) = \left\langle \frac{8}{17}, -\frac{2}{17} \right\rangle$.

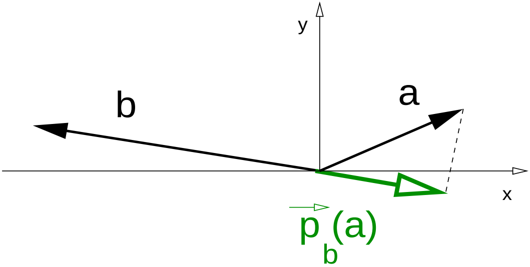
Example

Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Solution: The vector projection of **a** onto **b** is the vector

$$\mathbf{p}_b(a) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\right) \frac{\mathbf{b}}{|\mathbf{b}|} = \left(-\frac{2}{\sqrt{17}}\right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle.$$

We therefore obtain $\mathbf{p}_a(b) = \left\langle \frac{8}{17}, -\frac{2}{17} \right\rangle$.



Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.

Two main ways to introduce the cross product

Geometrical
definition

Two main ways to introduce the cross product

Geometrical definition $\longrightarrow$ Properties

Two main ways to introduce the cross product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.

Two main ways to introduce the cross product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.

Definition in components

Two main ways to introduce the cross product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.

Definition in components	$\longrightarrow$	Properties

Two main ways to introduce the cross product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.

Definition in components	$\longrightarrow$	Properties	$\longrightarrow$	Geometrical expression.

Two main ways to introduce the cross product

Geometrical definition $\longrightarrow$ Properties $\longrightarrow$ Expression in components.

Definition in components $\longrightarrow$ Properties $\longrightarrow$ Geometrical expression.

We choose the first way, like the textbook.

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- **Geometric definition of cross product.**
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.

Geometric definition of cross product

Definition

The *cross product* of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

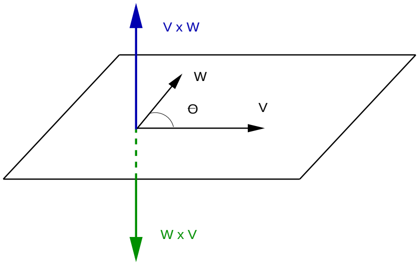
$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin(\theta).$$

Geometric definition of cross product

Definition

The *cross product* of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin(\theta).$$

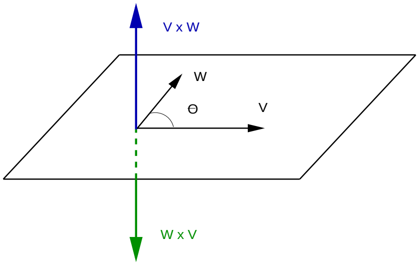


Geometric definition of cross product

Definition

The *cross product* of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin(\theta).$$



Remark: Cross product of two vectors is another vector; which is perpendicular to the original vectors.

Geometric definition of cross product

Theorem

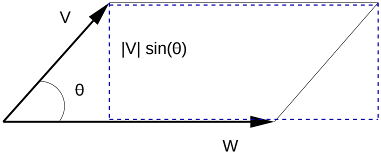
$|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$.

Geometric definition of cross product

Theorem

$|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by vectors* $\mathbf{v}$ *and* $\mathbf{w}$*.*

Proof.

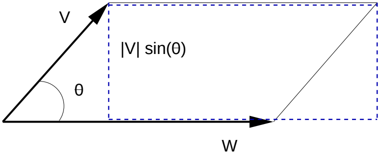


Geometric definition of cross product

Theorem

$|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by vectors* $\mathbf{v}$ *and* $\mathbf{w}$*.*

Proof.



The area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

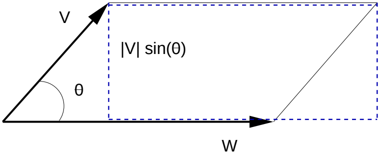
$$A = |\mathbf{w}|\big(|\mathbf{v}|\ \sin(\theta)\big)$$

Geometric definition of cross product

Theorem

$|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by vectors* $\mathbf{v}$ *and* $\mathbf{w}$.

Proof.



The area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

$$A = |\mathbf{w}|\big(|\mathbf{v}| \sin(\theta)\big) = |\mathbf{v} \times \mathbf{w}|.$$

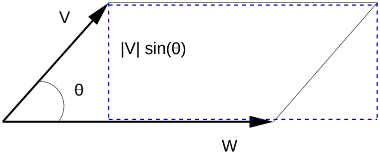
□

Geometric definition of cross product

Theorem

$|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by vectors* $\mathbf{v}$ *and* $\mathbf{w}$.

Proof.



The area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

$$A = |\mathbf{w}|\big(|\mathbf{v}|\,\sin(\theta)\big) = |\mathbf{v} \times \mathbf{w}|.$$

□

Definition

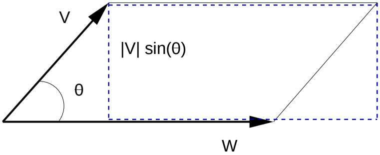
Two vectors are *parallel* iff the angle in between them is $\theta = 0$.

Geometric definition of cross product

Theorem

$|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by vectors* $\mathbf{v}$ *and* $\mathbf{w}$*.*

Proof.



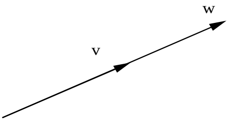
The area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

$$A = |\mathbf{w}|\big(|\mathbf{v}|\ \sin(\theta)\big) = |\mathbf{v} \times \mathbf{w}|.$$

□

Definition

Two vectors are *parallel* iff the angle in between them is $\theta = 0$.

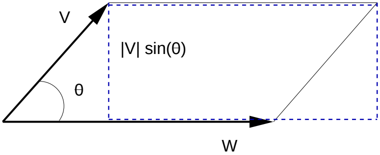


Geometric definition of cross product

Theorem

$|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$.

Proof.



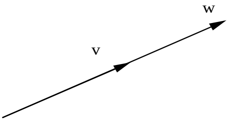
The area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

$$A = |\mathbf{w}|\big(|\mathbf{v}|\ \sin(\theta)\big) = |\mathbf{v} \times \mathbf{w}|.$$

□

Definition

Two vectors are *parallel* iff the angle in between them is $\theta = 0$.



Theorem

The non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel iff $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

Geometric definition of cross product

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

Geometric definition of cross product

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

Example

The closer the vectors **v**, **w** are to be parallel, the smaller is the area of the parallelogram they form,

Geometric definition of cross product

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

Example

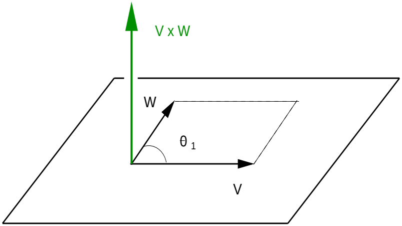
The closer the vectors $\mathbf{v}$, $\mathbf{w}$ are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$.

Geometric definition of cross product

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

Example

The closer the vectors $\mathbf{v}$, $\mathbf{w}$ are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$.

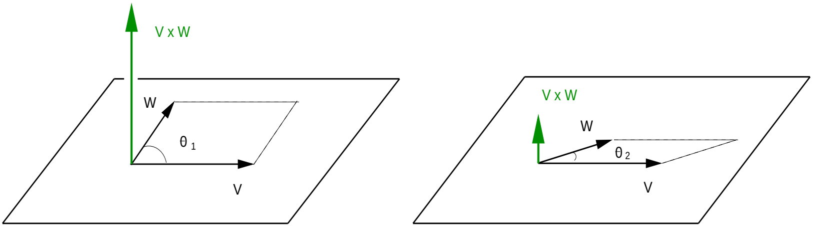


Geometric definition of cross product

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

Example

The closer the vectors $\mathbf{v}$, $\mathbf{w}$ are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$.



Geometric definition of cross product

Example

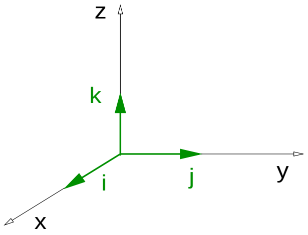
Compute all cross products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Geometric definition of cross product

Example

Compute all cross products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

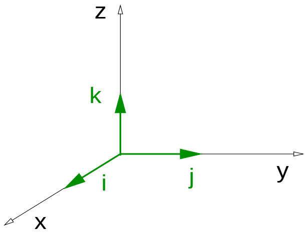


Geometric definition of cross product

Example

Compute all cross products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \qquad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \qquad \mathbf{k} \times \mathbf{k} = \mathbf{0},$$
$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}.$$

◁

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- **Properties of the cross product.**
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.

Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, *(skew-symmetric);*

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$, *(linear);*

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, *(linear);*

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, *(not associative).*

Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, *(skew-symmetric)*;

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$, *(linear)*;

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, *(linear)*;

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, *(not associative)*.

Proof.

Part (a) results from the right-hand rule

Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, *(skew-symmetric);*

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$, *(linear);*

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, *(linear);*

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, *(not associative).*

Proof.

Part (a) results from the right-hand rule and (b) from part (a).

Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, *(skew-symmetric)*;

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$, *(linear)*;

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, *(linear)*;

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, *(not associative)*.

Proof.

Part (a) results from the right-hand rule and (b) from part (a). Parts (b) and (c) are proven in a similar ways as the linear property of the dot product.

Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, *(skew-symmetric)*;

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$, *(linear)*;

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, *(linear)*;

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, *(not associative)*.

Proof.

Part (a) results from the right-hand rule and (b) from part (a). Parts (b) and (c) are proven in a similar ways as the linear property of the dot product. Part (d) is proven by giving an example. □

Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example.

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$.

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k})$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j})$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j})$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}\,) = -(\mathbf{i} \times \mathbf{j}\,) = -\mathbf{k}$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k}$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j}$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}\,) = -(\mathbf{i} \times \mathbf{j}\,) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j} \; = \mathbf{0}$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}\,) = -(\mathbf{i} \times \mathbf{j}\,) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j} \; = \mathbf{0} \quad \Rightarrow \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}.$$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}\,) = -(\mathbf{i} \times \mathbf{j}\,) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j} \; = \mathbf{0} \quad \Rightarrow \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}.$$

We conclude that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. $\triangleleft$

Properties of the cross product

Example

Show that the cross product is *not associative*, that is,
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \quad \Rightarrow \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}.$$

We conclude that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. $\lhd$

Recall: The cross product of parallel vectors vanishes.

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- **Cross product in vector components.**
- Determinants to compute cross products.
- Triple product and volumes.

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Proof: Use the cross product properties

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
Express $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$,

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1)\rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
Express $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$, then

$$\mathbf{v} \times \mathbf{w} = (v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}) \times (w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}).$$

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1)\rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
Express $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$, then

$$\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).$$

Use the linearity property.

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
Express $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$, then

$$\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).$$

Use the linearity property. The only non-zero terms involve $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and the symmetric analogues.

Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
Express $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$, then

$$\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).$$

Use the linearity property. The only non-zero terms involve $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and the symmetric analogues. The result is

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} + (v_3 w_1 - v_1 w_3)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}.$$

$\square$

Cross product in vector components.

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Cross product in vector components.

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \big\langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \big\rangle$$

Cross product in vector components.

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \big\langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \big\langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \big\rangle$$

Cross product in vector components.

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \Big\langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \Big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \Big\langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \Big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle$$

Cross product in vector components.

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \big\langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \big\langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle \quad \Rightarrow \quad \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle.$$

◁

Cross product in vector components.

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \big\langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \big\langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle \quad \Rightarrow \quad \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle.$$

$\triangleleft$

Exercise: Find the angle between $\mathbf{v}$ and $\mathbf{w}$ above, using both the cross and the dot products. Verify that you get the same answer.

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- **Determinants to compute cross products.**
- Triple product and volumes.

Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

Recall:

(a) The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

Recall:

(a) The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(b) The determinant of a 3×3 matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

Recall:

(a) The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(b) The determinant of a 3×3 matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

2×2 determinants are used to find 3×3 determinants.

Determinants to compute cross products.

Theorem

The formula to compute determinants of 3×3 *matrices can be used to find the the cross product* $\mathbf{v} \times \mathbf{w}$, *where* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *as follows*

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Determinants to compute cross products.

Theorem

The formula to compute determinants of 3×3 *matrices can be used to find the the cross product* $\mathbf{v} \times \mathbf{w}$, *where* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *as follows*

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof: Indeed, a straightforward computation shows that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Determinants to compute cross products.

Theorem

The formula to compute determinants of 3×3 *matrices can be used to find the the cross product* $\mathbf{v} \times \mathbf{w}$, *where* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *as follows*

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof: Indeed, a straightforward computation shows that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\,\mathbf{i}$$

Determinants to compute cross products.

Theorem

The formula to compute determinants of 3×3 *matrices can be used to find the the cross product* $\mathbf{v} \times \mathbf{w}$, *where* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *as follows*

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof: Indeed, a straightforward computation shows that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} - (v_1 w_3 - v_3 w_1)\,\mathbf{j}$$

Determinants to compute cross products.

Theorem

The formula to compute determinants of 3×3 matrices can be used to find the the cross product $\mathbf{v} \times \mathbf{w}$, where $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, as follows

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof: Indeed, a straightforward computation shows that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} - (v_1 w_3 - v_3 w_1)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}.$$

$\square$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$\mathbf{w} \times \mathbf{v}$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$\mathbf{w} \times \mathbf{v} = (9 - 2)\, \mathbf{i}$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9-2)\,\mathbf{i} - (-6-1)\,\mathbf{j}$$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9-2)\,\mathbf{i} - (-6-1)\,\mathbf{j} + (-4-3)\,\mathbf{k}$$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9-2)\,\mathbf{i} - (-6-1)\,\mathbf{j} + (-4-3)\,\mathbf{k} \quad \Rightarrow \quad \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.$$

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9-2)\,\mathbf{i} - (-6-1)\,\mathbf{j} + (-4-3)\,\mathbf{k} \quad \Rightarrow \quad \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.$$

The properties of the determinant imply $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9-2)\,\mathbf{i} - (-6-1)\,\mathbf{j} + (-4-3)\,\mathbf{k} \quad \Rightarrow \quad \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.$$

The properties of the determinant imply $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.
Hence, $\mathbf{v} \times \mathbf{w} = \langle -7, -7, 7 \rangle$. $\triangleleft$

Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- **Triple product and volumes.**

Triple product and volumes

Definition

The *triple product* of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, is the scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Triple product and volumes

Definition

The *triple product* of the vectors **u**, **v** , **w**, is the scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Remarks:

(a) The triple product of three vectors is a scalar.

Triple product and volumes

Definition

The *triple product* of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, is the scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Remarks:

(a) The triple product of three vectors is a scalar.

(b) The parentheses are important.

Triple product and volumes

Definition

The *triple product* of the vectors **u**, **v** , **w**, is the scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Remarks:

(a) The triple product of three vectors is a scalar.

(b) The parentheses are important. First do the cross product,

Triple product and volumes

Definition

The *triple product* of the vectors **u**, **v**, **w**, is the scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Remarks:

(a) The triple product of three vectors is a scalar.

(b) The parentheses are important. First do the cross product, and only then dot the resulting vector with the first vector.

Triple product and volumes

Definition

The *triple product* of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, is the scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Remarks:

(a) The triple product of three vectors is a scalar.

(b) The parentheses are important. First do the cross product, and only then dot the resulting vector with the first vector.

Theorem (Cyclic rotation formula for triple product)

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}).$$

Triple product and volumes

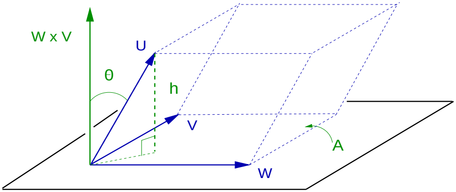
Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Triple product and volumes

Theorem

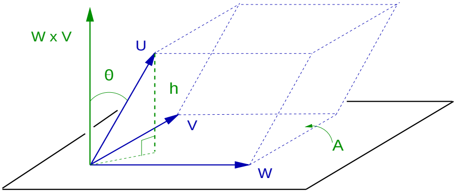
The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Triple product and volumes

Theorem

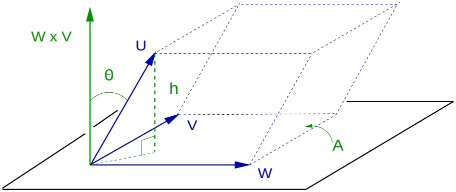
The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$.

Triple product and volumes

Theorem
The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



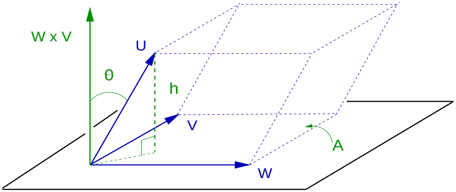
Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$. Then,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

Triple product and volumes

Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



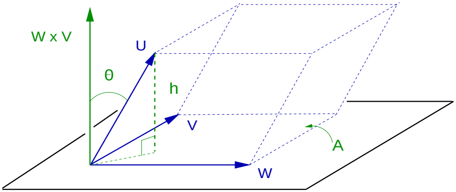
Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$. Then,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,|\cos(\theta)|$$

Triple product and volumes

Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



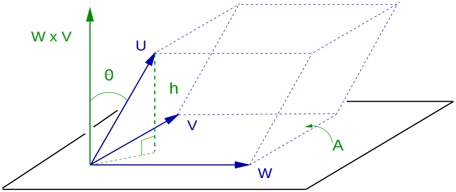
Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$. Then,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,|\cos(\theta)| = h\,|\mathbf{v} \times \mathbf{w}|.$$

Triple product and volumes

Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$. Then,

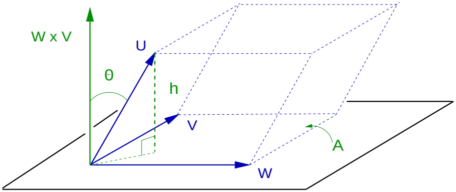
$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,|\cos(\theta)| = h\,|\mathbf{v} \times \mathbf{w}|.$$

$|\mathbf{v} \times \mathbf{w}|$ is the area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$.

Triple product and volumes

Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$. Then,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,|\cos(\theta)| = h\,|\mathbf{v} \times \mathbf{w}|.$$

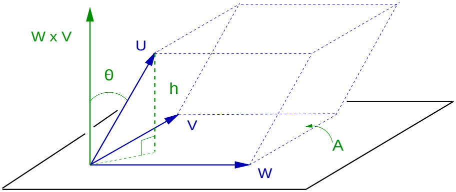
$|\mathbf{v} \times \mathbf{w}|$ is the area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$. So,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = h\,A,$$

Triple product and volumes

Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.



Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos(\theta)$. Then,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,|\cos(\theta)| = h\,|\mathbf{v} \times \mathbf{w}|.$$

$|\mathbf{v} \times \mathbf{w}|$ is the area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$. So,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = h\,A,$$

which is the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. □

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w}$$

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2 + 2)\,\mathbf{i} - (3 - 1)\,\mathbf{j} + (-6 - 2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$.

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle$$

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,$$

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -24$.

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -24$. We conclude that $V = 24$. $\triangleleft$

The triple product and volumes

Remark: The triple product can be computed with a determinant.

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3\rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle 3, 2, 1\rangle$, $\mathbf{w} = \langle 1, -2, 1\rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (1)(2+2) - (2)(3-1) + (3)(-6-2),$$

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (1)(2+2) - (2)(3-1) + (3)(-6-2),$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4 - 4 - 24$

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (1)(2+2) - (2)(3-1) + (3)(-6-2),$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4 - 4 - 24 = -24$.

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (1)(2+2) - (2)(3-1) + (3)(-6-2),$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4 - 4 - 24 = -24$. Hence $V = 24$. $\lhd$