Name:

TEST 3

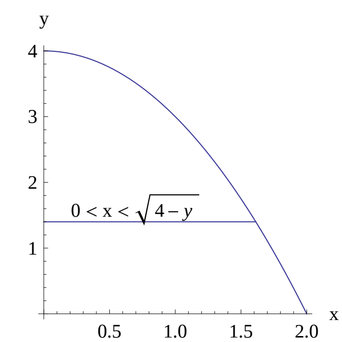
No Calculators

1:
2:
3:
4:
5:
6:

1 (20 points) Let $f(x,y) = 8x^3 + y^3 + 6xy$. Find and classify all critical points.

$f_x = 24x^2 + 6y,\ f_{xx} = 48x,\ f_{xy} = 6$
$f_y = 3y^2 + 6x,\ f_{yy} = 6y$
Critical points when: $24x^2 + 6y = 0$ and $3y^2 + 6x = 0$ or equivalently
$y = -4x^2$ and $x(8x^3 + 1) = 0$, hence, $x = 0$ or $x = -1/2$.
We have two critical points: $(0,0)$ and $(-1/2,-1)$.
At $(0,0)$ we have $f_{xx}f_{yy} - f_{xy}^2 = -36$ hence $(0,0)$ is a saddle point.
At $(-1/2,-1)$ we have $f_{xx}f_{yy} - f_{xy}^2 = 3 * 36 > 0$ and $f_{xx} = -24 < 0$ hence
$(-1/2,-1)$ is a local maximum.

2 (20 points) Evaluate $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx.$



$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy = \frac{1}{2}\int_0^4 e^{2y} dy = \frac{e^8 - 1}{4}$$

3 (20 points) Find the average value of $1/(x^2 + y^2 + 1)$ within a disk $x^2 + y^2 \leq 1$.

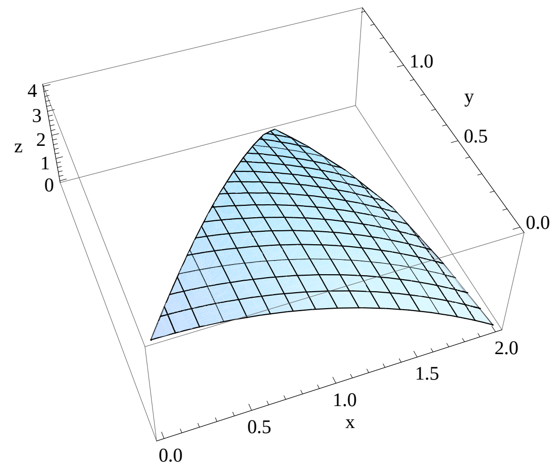
area $= \pi$

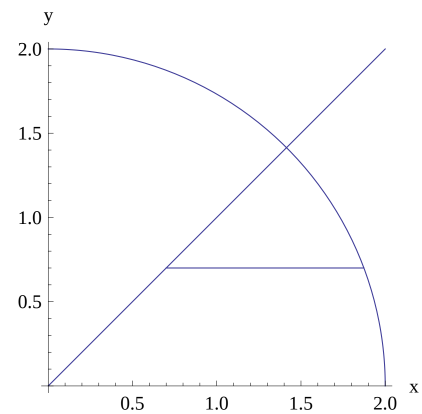
$$\text{average value } = \frac{1}{\text{area}} \int\int \frac{dA}{x^2 + y^2 + 1} = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \frac{r\, d\theta\, dr}{r^2 + 1} = \int_0^1 \frac{2r\, dr}{r^2 + 1} = \ln 2$$

4 (20 points) Find the volume of the region in the first octant that is bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the plane $z = 0$ and on the sides by the planes $y = x$ and $y = 0$.

$$V = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^{4-x^2-y^2} dz\, dx\, dy = \int_0^2 \int_0^{\pi/4} \int_0^{4-r^2} r\, dz\, d\theta\, dr$$

$$V = \frac{\pi}{4}\int_0^2 (4r - r^3) dr = \pi.$$





5 (20 points) Find the volume of the cone $\phi = \pi/3$ inside the sphere $\rho \leq 1$.

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \, d\phi \, d\theta = \frac{1}{6} \int_0^{2\pi} d\theta = \frac{\pi}{3}$$

6 (10 points) (extra credit points only if you get 90 or more points on problems 1-5) Define

$$I(a) = \int_{-a}^{a} e^{-x^2} \, dx \quad \text{for } a > 0.$$

a) (2 points) Show that $\int_{-a}^{a} \int_{-a}^{a} e^{-x^2-y^2} \, dx \, dy = I(a)^2$.

b) (4 points) Evaluate $J(a) = \int \int_{x^2+y^2 \leq a^2} e^{-x^2-y^2} \, dx \, dy$.

c) (2 points) Show that $J(a) < I(a)^2 < J(a\sqrt{2})$.

d) (2 points) Find the limit of $I(a)$ as $a \to \infty$ and hence the value of $\int_{-\infty}^{\infty} e^{-x^2} \, dx$.

a) $\int_{-a}^{a} \int_{-a}^{a} e^{-x^2-y^2} \, dx \, dy = \int_{-a}^{a} I(a) e^{-y^2} \, dy = I(a)^2$

b) $J(a) = \int_0^{2\pi} \int_0^{a} e^{-r^2} r \, dr \, d\theta = (1 - e^{-a^2})\pi$.

c) Circle of radius a is contained in a square with side $2a$ which is contained in a circle of radius $a\sqrt{2}$. This, positivity of the integrand and 3 and 4 on p.1078 imply $J(a) < I(a)^2 < J(a\sqrt{2})$.

d) Since $J(a) \to \pi$ as $a \to \infty$, the lower bound and the upper bound of $I(a)^2$ converge to π. Hence

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$