Name:

TEST 2

No Calculators

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1 (16 points) A particle's acceleration at time t is given by

$$\frac{d^2\mathbf{r}}{dt^2}(t) = < -\sin t - \cos t, \cos t - \sin t, 0 >,$$

$$\text{initial velocity } \frac{d\mathbf{r}}{dt}(0) = < 1, 1, 1 >, \qquad \text{initial position } \mathbf{r}(0) = < 0, 0, 0 > .$$

Find the particle's position $\mathbf{r}(t)$ at time t and the arc length of its trajectory from time $t = 0$ to $t = 1$.

Integration gives velocity $\mathbf{r}' = < \cos t - \sin t, \sin t + \cos t, 0 > +\mathbf{c}$ and $\mathbf{c}$ needs to be such that the initial velocity is correct. Hence $\mathbf{r}' = < \cos t - \sin t, \sin t + \cos t, 1 >$. Integration gives position $\mathbf{r} = < \sin t + \cos t, -\cos t + \sin t, t > +\mathbf{c}_1$ and $\mathbf{c}_1$ needs to be such that the initial position is correct. Hence

$$\mathbf{r}(t) = < \sin t + \cos t - 1, 1 - \cos t + \sin t, t > .$$

Since $|\mathbf{r}'|^2 = (\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1 = 3$ we have that the arc length equals

$$\int_0^1 |\mathbf{r}'(t)| dt = \sqrt{3}.$$

2 (14 points) Does $f(x, y) = xy/(x - y)$ have a limit as (x, y) approaches $(0, 0)$? Justify your answer.

No.
Solving $f(x, y) = 1$ gives $y = x/(1 + x)$, hence $f(x, x/(1 + x)) = 1$ as $x \to 0$.
Solving $f(x, y) = -1$ gives $y = x/(1 - x)$, hence $f(x, x/(1 - x)) = -1$ as $x \to 0$.
By the Two-Path Test the limit does not exist.

3 (14 points) Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3 x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Differentiation gives $y + 3z^2 z_x x + z^3 - 2y z_x = 0$ hence $z_x = -2$.

4 (14 points) Let $w = f(r, \phi)$, where r and ϕ are the polar coordinates, i.e. $x = r\cos\phi$ and $y = r\sin\phi$. Express w_x as a function of r and ϕ.

$r = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}(y/x)$
$r_x = x/r = \cos\phi$, $\phi_x = -y/(x^2 + y^2) = -(\sin\phi)/r$
$w_x = f_r r_x + f_\phi \phi_x = f_r \cos\phi - f_\phi(\sin\phi)/r$.

5 (14 points) Find the equations of tangent plane and of the normal line to the surface $z = x^2 - y^2$ at the point $(2, -1, 3)$.

$f = x^2 - y^2 - z = 0$
$f_x = 2x = 4,\ f_y = -2y = 2,\ f_z = -1,\ \nabla f = < 4, 2, -1 >$
Tangent plane: $4(x-2) + 2(y+1) - (z-3) = 0$
Normal line: $x = 2 + 4t,\ y = -1 + 2t,\ z = 3 - t$

6 (14 points) Let $f(x, y, z) = x/y - yz$. Give a good estimate of the maximum increase of f as we move a distance 0.01 from the point $(1, 1, -1)$.

$f_x = 1/y = 1,\ f_y = -x/y^2 - z = 0,\ f_z = -1$
$\nabla f = < 1, 0, -1 >,\ |\nabla f| = \sqrt{2}$
f increases the most in the direction $\mathbf{u} = \nabla f / |\nabla f|$

$$\left.\frac{df}{ds}\right|_{\mathbf{u}} = \nabla f \cdot \mathbf{u} = |\nabla f|$$

$df = |\nabla f| ds = 0.01\sqrt{2}$

7 (14 points) Let $f(x, y) = e^{\frac{x^2}{y} - 1} \cos(x^2 - y^3)$. Find the linearization of f at the point $(1, 1)$.

$$f_x = \frac{2x}{y} e^{\frac{x^2}{y} - 1} \cos(x^2 - y^3) - e^{\frac{x^2}{y} - 1} \sin(x^2 - y^3)\ (2x) = 2$$

$$f_y = -\frac{x^2}{y^2} e^{\frac{x^2}{y} - 1} \cos(x^2 - y^3) - e^{\frac{x^2}{y} - 1} \sin(x^2 - y^3)\ (-3y^2) = -1$$

Linearization:
$L(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) = 1 + 2(x - 1) - (y - 1) = 2x - y$

8 (10 points) (extra credit points only if you get 90 or more points on problems 1-7) The difference $\mathbf{r}$ between the positions of two bodies moving in their gravitational field can be rescaled to satisfy

$$\mathbf{r}'' = -\frac{\mathbf{r}}{|\mathbf{r}|^3}.$$

Show that $\mathbf{r}' \times \mathbf{r}$ is a constant.

$(\mathbf{r}' \times \mathbf{r})' = \mathbf{r}'' \times \mathbf{r} + \mathbf{r}' \times \mathbf{r}' = \mathbf{r}'' \times \mathbf{r} = -|\mathbf{r}|^{-3} \mathbf{r} \times \mathbf{r} = 0$