Math 234, Practice Test #2

Show your work in all the problems.

1. In what directions is the derivative of

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

at $P = (1,1)$ equal to zero ?

2. Find an equation for the level surface of the function through the given point P.

(a)

$$f(x,y,z) = z - x^2 - y^2 \;,\; P = (3,-1,1)$$

(b)

$$f(x,y,z) = \int_x^y \frac{dt}{\sqrt{1-t^2}} + \int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2-1}} \;,\; P = (-1,1/2,1)$$

3. Compute the limits of the following expressions if they exist. If you think they don't, consider different paths of approach to show that they do not.

(a)

$$\frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1} \;,\; (x,y) \to (4,3) \,,\; x \neq y+1$$

(b)

$$\frac{x^2 + y^2}{xy} \;,\; (x,y) \to (0,0) \,,\; xy \neq 0$$

4. Compute all second order partial derivatives of the function

$$f(x,y) = x \sin y + y \sin x + xy$$

5. Find $\frac{dw}{dt}$ if $w = \sin(xy + \pi)$, $x = e^t$ and $y = \ln(t+1)$. Then evaluate at $t = 0$.

Solutions

1. We first compute the gradient vector of f:

$$\nabla f = \left(\frac{4xy^2}{(x^2+y^2)^2}, -\frac{4x^2y}{(x^2+y^2)^2} \right)$$

Evaluating at $(1,1)$ yields

$$\nabla f(1,1) = (1,-1).$$

The directions $\mathbf{u}$ in which the directional derivative $(D_{\mathbf{u}}f)(1,1)$ is zero are the unit vectors orthogonal to $\nabla f(1,1) = (1,-1)$, i.e. $\mathbf{u} = (u_1, u_2)$ has to satisfy

$$\nabla f(1,1) \bullet \mathbf{u} = (1,-1) \bullet (u_1, u_2) = u_1 - u_2 = 0$$

and $u_1^2 + u_2^2 = 1$. We get

$$\mathbf{u} = \frac{1}{\sqrt{2}}(1,1) \quad \text{and} \quad -\mathbf{u} = -\frac{1}{\sqrt{2}}(1,1).$$

2. (a) The level surfaces are given by the equations

$$c = z - x^2 - y^2$$

where c is a constant. In order to single out the one which passes through the point P we need to insert the coordinates of P and compute the constant c. Hence

$$c = 1 - 3^2 - (-1)^2 = -9$$

so that the desired equation is $-9 = z - x^2 - y^2$.

(b) We compute the integrals first

$$\int_x^y \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(y) - \sin^{-1}(x)$$

and

$$\int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2-1}} = \sec^{-1}|z| - \sec^{-1}(\sqrt{2}) = \sec^{-1}|z| - \frac{\pi}{4}$$

The latter follows from the fact that $\sec^{-1}(\sqrt{2})$ is the 'angle' θ where $\cos\theta = 1/\sqrt{2}$ which is $\pi/4$. We insert now $x = -1$, $y = 1/2$ and $z = 1$:

$$\begin{aligned} c &= f(-1, 1/2, 1) \\ &= \sin^{-1}(1/2) - \sin^{-1}(-1) + \sec^{-1}|1| - \frac{\pi}{4} \\ &= \frac{\pi}{6} + \frac{\pi}{2} + 0 - \frac{\pi}{4} \\ &= \frac{5\pi}{12} \end{aligned}$$

The equation of the level surface is then

$$\frac{5\pi}{12} = \sin^{-1}(y) - \sin^{-1}(x) + \sec^{-1}|z| - \frac{\pi}{4}$$

or

$$\frac{2\pi}{3} = \sin^{-1}(y) - \sin^{-1}(x) + \sec^{-1}|z|$$

3. (a) We use the following formula to simplify the given expression

$$(\sqrt{x} - \sqrt{y+1})(\sqrt{x} + \sqrt{y+1}) = (\sqrt{x})^2 - (\sqrt{y+1})^2 = x - (y+1) = x - y - 1$$

Then

$$\frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1} = \frac{1}{\sqrt{x} + \sqrt{y+1}}$$

and the limit can simply be computed by inserting $(x, y) = (4, 3)$ which yields $1/4$.

(b) No further simplification as in problem (a) is possible here. We look at the level curves

$$c = \frac{x^2 + y^2}{xy} \text{ or } x^2 + y^2 - cxy = 0$$

Solving this for y (quadratic equation) we get

$$y = \frac{1}{2}(c \pm \sqrt{c^2 - 4})\, x = kx.$$

Note that all these curves pass through the origin, so there is no limit for $(x, y) \to (0, 0)$. In order to confirm, we insert $y = kx$ and we get

$$\frac{x^2+y^2}{xy} = \frac{x^2+k^2x^2}{kx^2} = \frac{1+k^2}{k}$$

Taking the limit $x \to 0$ still yields $(1 + k^2)/k$, i.e. the value depends on the direction in which we approach the origin.

4. We have

$$f(x, y) = x \sin y + y \sin x + xy$$

The first order derivatives are given by

$$f_x = \sin y + y \cos x + y \ , \ f_y = x \cos y + \sin x + x$$

Then

$$f_{xx} = -y \sin x \ , \ f_{yy} = -x \sin y \ , \ f_{xy} = f_{yx} = \cos y + \cos x + 1$$

5. We compute

$$\frac{\partial w}{\partial x} = y \cos(xy + \pi) \ , \ \frac{\partial w}{\partial y} = x \cos(xy + \pi)$$

and

$$x'(t) = e^t \ , \ y'(t) = \frac{1}{t+1}.$$

Then

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial y} y'(t) \\ &= e^t \ln(t+1) \cos(e^t \ln(t+1) + \pi) + \frac{e^t}{t+1} \cos(e^t \ln(t+1) + \pi) \\ &= \left(\ln(t+1) + \frac{1}{t+1} \right) e^t \cos(e^t \ln(t+1) + \pi) \end{aligned}$$

Evaluating at $t = 0$ yields

$$\left. \frac{dw}{dt} \right|_{t=0} = \cos(\pi) = -1$$