Math 234, Practice Test #1

Show your work in all the problems.

1. Find parametric equations for the line in which the planes $x+2y+z=1$ and $x-y+2z=-8$ intersect.

2. Compute the distance from the point $(2,2,3)$ to the plane through the points $A=(0,0,0)$, $B=(2,0,-1)$ and $C=(2,-1,0)$.

3. Compute the area of the parallelogram with three of its vertices given by
$$A=(2,-2,1)\ ,\ B=(3,-1,2) \text{ and } C=(3,-1,1)$$

4. *(Cancellation in a dot product ?)* Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors with $\mathbf{u} \neq 0$. Is it true that $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w}$ implies $\mathbf{v} = \mathbf{w}$? If you think it is true explain why, otherwise provide a counterexample.

5. Sketch the surface given by the equation $z = 1 - x^2$.

6. Describe the given sets with a single equation or a pair of equations: The circle of radius 1 centered at $(-3,4,1)$ and lying in a plane parallel to the

   (a) xy-plane

   (b) yz-plane

   (c) xz-plane

Solutions

1. Normal vectors of the two planes are given by

$$\mathbf{n}_1 = (1, 2, 1) \text{ and } \mathbf{n}_2 = (1, -1, 2)$$

respectively. The line of intersection is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. The following vector is then parallel to the line of intersection:

$$\begin{aligned}
\mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \mathbf{k} \\
&= 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} \\
&= (5, -1, -3)
\end{aligned}$$

We also need to find a point which lies on the line of intersection. We insert $x = 1 - 2y - z$ (which is derived from the first equation) into the second equation, and we get

$$-8 = x - y + 2z = 1 - 2y - z - y + 2z = 1 - 3y + z$$

as well as $z = -9 + 3y$ and $x = 1 - 2y - z = 10 - 5y$. This means we can pick any value we want for y and compute x, z using the previous formulas. For $y = 0$ we get $x = 10$ and $z = -9$ so that the point $(10, 0, -9)$ lies on the line of intersection. Parametric equations of the line are then given by

$$x = 10 + 5t\,,\; y = -t\,,\; z = -9 - 3t$$

Comment: There are many possible solutions which are all correct (and for which you would get full credit). For example, the vector $(-5, 1, 3)$ is also parallel to the line of intersection (do you know why ?). Later on, instead of choosing $y = 0$ to find a point on the line we could have chosen something else, for example $y = 1$. We would have obtained $x = 10 - 5y = 5$ and $z = -9 + 3y = -6$, so that we use the point $(5, 1, -6)$ instead. Remember that there are many points on a line, *and there is no reason to prefer one over the other. The parametric equations would then look as follows*

$$x = 5 - 5t\,,\; y = 1 + t\,,\; z = -6 + 3t.$$

Although these equations are different they describe the same line, and they are therefore also a valid solution to the problem.

2. We need to find a normal vector $\mathbf{n}$ to the plane, i.e. a vector perpendicular to both $\vec{AB} = (2, 0, -1)$ and $\vec{AC} = (2, -1, 0)$. We get a such a vector by taking the cross product

$$\mathbf{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

If P is any point in the plane then the distance d between $S = (2, 2, 3)$ and the plane is given by

$$d = \frac{|\,\vec{PS} \bullet \mathbf{n}\,|}{|\mathbf{n}|}$$

The easiest pick for P is probably $(0, 0, 0)$, so that $\vec{PS} = (2, 2, 3)$. We get

$$\vec{PS} \bullet \mathbf{n} = (2, 2, 3) \bullet (-1, -2, -2) = -2 - 4 - 6 = -12$$

and

$$|\mathbf{n}| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

so that

$$d = \frac{|-12|}{3} = 4.$$

3. The area of the parallelogram is given by

$$|\,\vec{AB} \times \vec{AC}\,|$$

We have

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} = (-1, 1, 0)$$

and

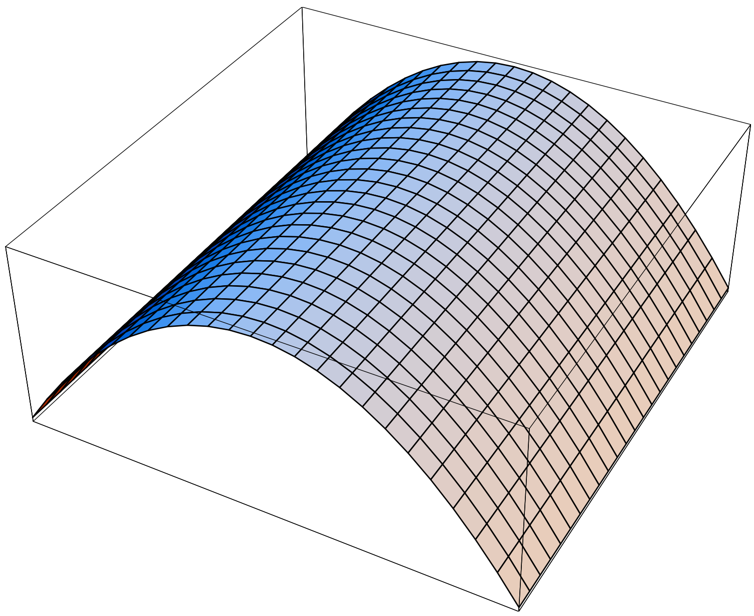
$$|\,\vec{AB} \times \vec{AC}\,| = \sqrt{2}.$$

4. The statement is false. Pick two different vectors $\mathbf{v}$ and $\mathbf{w}$, for example $\mathbf{v} = (1, 0, 0)$ and $\mathbf{w} = (0, 1, 0)$. Then choose $\mathbf{u}$ perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, for example $\mathbf{u} = (0, 0, 1)$ would do. Then

$$\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w} = 0$$

but $\mathbf{v} \neq \mathbf{w}$.

5. The surface is a cylinder over the parabola $z = 1 - x^2$ in the xz-plane:



6. (a) $(x + 3)^2 + (y - 4)^2 = 1 \ , \ z = 1$

   (b) $(y - 4)^2 + (z - 1)^2 = 1 \ , \ x = -3$

   (c) $(x + 3)^2 + (z - 1)^1 = 1 \ , \ y = 4$