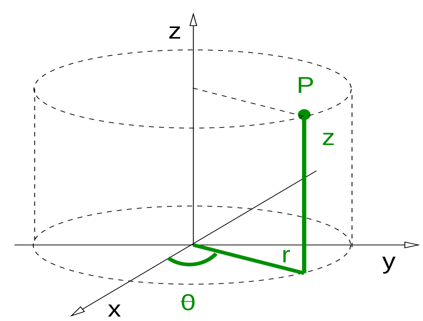
Integrals in cylindrical, spherical coordinates (Sect. 15.7)

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - Spherical coordinates in space.
  - Triple integral in spherical coordinates.

Cylindrical coordinates in space.

Definition
The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (r, θ, z) defined by the picture.



Remark: Cylindrical coordinates are just polar coordinates on the plane $z = 0$ together with the vertical coordinate z.

Theorem (Cartesian-cylindrical transformations)
The Cartesian coordinates of a point $P = (r, \theta, z)$ are given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.

The cylindrical coordinates of a point $P = (x, y, z)$ in the first and fourth quadrant are $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and $z = z$.

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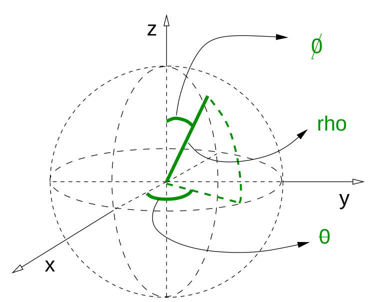
Spherical coordinates in $\mathbb{R}^3$



Definition

The *spherical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (ρ, ϕ, θ) defined by the picture.

Theorem (Cartesian-spherical transformations)

The Cartesian coordinates of $P = (\rho, \phi, \theta)$ *in the first quadrant are given by* $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, *and* $z = \rho \cos(\phi)$.

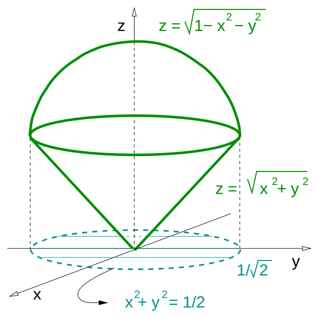
The spherical coordinates of $P = (x, y, z)$ *in the first quadrant are* $\rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arctan\Big(\dfrac{y}{x}\Big)$, *and* $\phi = \arctan\Big(\dfrac{\sqrt{x^2+y^2}}{z}\Big)$.

Spherical coordinates in $\mathbb{R}^3$

Example

Use spherical coordinates to express region between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: ($x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, $z = \rho \cos(\phi)$.)



The top surface is the sphere $\rho = 1$.
The bottom surface is the cone:

$$\rho \cos(\phi) = \sqrt{\rho^2 \sin^2(\phi)}$$

$$\cos(\phi) = \sin(\phi),$$

so the cone is $\phi = \dfrac{\pi}{4}$.

Hence: $R = \Big\{(\rho, \phi, \theta) \ : \ \theta \in [0, 2\pi], \ \phi \in \Big[0, \dfrac{\pi}{4}\Big], \ \rho \in [0, 1]\Big\}$.

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Triple integral in spherical coordinates

Theorem

If the function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in spherical coordinates as follows,

$$\iiint_R f\,dv = \iiint_R f(\rho,\phi,\theta)\,\rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta.$$

Remark:

- Spherical coordinates are useful when the integration region R is described in a simple way using spherical coordinates.
- Notice the extra factor $\rho^2 \sin(\phi)$ on the right-hand side.

Triple integral in spherical coordinates

Example

Find the volume of a sphere of radius R.

Solution: Sphere: $S = \{\theta \in [0, 2\pi],\ \phi \in [0, \pi],\ \rho \in [0, R]\}$.

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin(\phi)\,d\rho\,d\phi\,d\theta,$$

$$V = \Big[\int_0^{2\pi} d\theta\Big]\Big[\int_0^{\pi} \sin(\phi)\,d\phi\Big]\Big[\int_0^{R} \rho^2\,d\rho\Big],$$

$$V = 2\pi\Big[-\cos(\phi)\Big|_0^{\pi}\Big]\frac{R^3}{3},$$

$$V = 2\pi\big[-\cos(\pi)+\cos(0)\big]\frac{R^3}{3};$$

hence: $V = \frac{4}{3}\pi R^3$. ◁

Triple integral in spherical coordinates

Example

Use spherical coordinates to find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.

Solution: $R = \Big\{(\rho, \phi, \theta) \ : \ \theta \in [0, 2\pi], \ \phi \in \Big[0, \frac{\pi}{4}\Big], \ \rho \in [0, 1]\Big\}.$

The calculation is simple, the region is a simple section of a sphere.

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta,$$

$$V = \Big[\int_0^{2\pi} d\theta\Big] \Big[\int_0^{\pi/4} \sin(\phi) \, d\phi\Big] \Big[\int_0^1 \rho^2 \, d\rho\Big],$$

$$V = 2\pi \Big[-\cos(\phi)\Big|_0^{\pi/4}\Big] \Big(\frac{\rho^3}{3}\Big|_0^1\Big),$$

$$V = 2\pi \Big[-\frac{\sqrt{2}}{2} + 1\Big] \frac{1}{3} \quad \Rightarrow \quad V = \frac{\pi}{3}(2 - \sqrt{2}). \qquad \lhd$$

Triple integral in spherical coordinates

Example

Find the integral of $f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$ in the region $R = \{x \geqslant 0, \ y \geqslant 0, \ z \geqslant 0, \ x^2 + y^2 + z^2 \leqslant 1\}$ using spherical coordinates.

Solution: $R = \Big\{\theta \in \Big[0, \frac{\pi}{2}\Big], \ \phi \in \Big[0, \frac{\pi}{2}\Big], \ \rho \in [0, 1]\Big\}$. Hence,

$$I = \iiint_R f \, dv = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 e^{\rho^3} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta,$$

$$I = \Big[\int_0^{\pi/2} d\theta\Big] \Big[\int_0^{\pi/2} \sin(\phi) \, d\phi\Big] \Big[\int_0^1 e^{\rho^3} \rho^2 \, d\rho\Big].$$

Use substitution: $u = \rho^3$, hence $du = 3\rho^2 \, d\rho$, so

$$I = \frac{\pi}{2} \Big[-\cos(\phi)\Big|_0^{\frac{\pi}{2}}\Big] \int_0^1 \frac{e^u}{3} \, du \ \Rightarrow \ \iiint_R f \, dv = \frac{\pi}{6}(e - 1). \quad \lhd$$

Triple integral in spherical coordinates

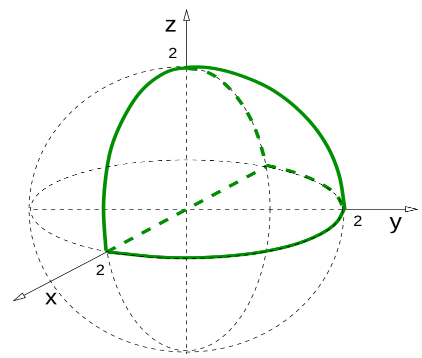
Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

Solution: ($x = \rho \sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.)

- Limits in x: $|x| \leqslant 2$;
- Limits in y: $0 \leqslant y \leqslant \sqrt{4-x^2}$, so the positive side of the disk $x^2+y^2 \leqslant 4$.
- Limits in z: $0 \leqslant z \leqslant \sqrt{4-x^2-y^2}$, so a positive quarter of the ball $x^2+y^2+z^2 \leqslant 4$.



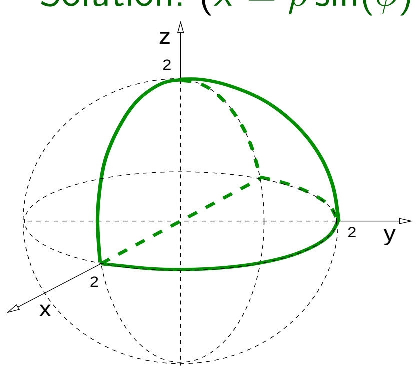
Triple integral in spherical coordinates

Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

Solution: ($x = \rho \sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.)



- Limits in θ: $\theta \in [0.\pi]$;
- Limits in ϕ: $\phi \in [0, \pi/2]$;
- Limits in ρ: $\rho \in [0, 2]$.
- The function to integrate is: $f = \rho^2 \sin(\phi)\sin(\theta)$.

$$I = \int_0^{\pi} \int_0^{\pi/2} \int_0^2 \rho^2 \sin(\phi)\sin(\theta)\big(\rho^2\sin(\phi)\big)\, d\rho\, d\phi\, d\theta.$$

Triple integral in spherical coordinates

Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

Solution: $I = \int_0^{\pi} \int_0^{\pi/2} \int_0^2 \rho^2 \sin(\phi)\sin(\theta)\big(\rho^2 \sin(\phi)\big)\, d\rho\, d\phi\, d\theta.$

$$I = \Big[\int_0^{\pi} \sin(\theta)\, d\theta\Big]\Big[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\Big]\Big[\int_0^2 \rho^4\, d\rho\Big],$$

$$I = \Big(-\cos(\theta)\Big|_0^{\pi}\Big)\Big[\int_0^{\pi/2} \frac{1}{2}\big(1-\cos(2\phi)\big)\, d\phi\Big]\Big(\frac{\rho^5}{5}\Big|_0^2\Big),$$

$$I = 2\,\frac{1}{2}\Big[\Big(\frac{\pi}{2}-0\Big) - \frac{1}{2}\Big(\sin(2\phi)\Big|_0^{\pi/2}\Big)\Big]\frac{2^5}{5} \quad \Rightarrow \quad I = \frac{2^4\pi}{5}. \quad \triangleleft$$

Triple integral in spherical coordinates

Example

Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^{2} 3\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$

Solution: Recall: $\sec(\phi) = 1/\cos(\phi)$.

$$I = 2\pi \int_0^{\pi/3} \Big(\rho^3\Big|_{\sec(\phi)}^{2}\Big) \sin(\phi)\, d\phi,$$

$$I = 2\pi \int_0^{\pi/3} \Big(2^3 - \frac{1}{\cos^3(\phi)}\Big) \sin(\phi)\, d\phi$$

In the second term substitute: $u = \cos(\phi)$, $du = -\sin(\phi)\, d\phi$.

$$I = 2\pi\Big[2^3\Big(-\cos(\phi)\Big|_0^{\pi/3}\Big) + \int_1^{1/2} \frac{du}{u^3}\Big].$$

Triple integral in spherical coordinates

Example

Compute the integral $I = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec(\phi)}^{2} 3\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$

Solution: $I = 2\pi\Big[2^3\Big(-\cos(\phi)\Big|_0^{\pi/3}\Big) + \int_1^{1/2}\frac{du}{u^3}\Big].$

$$I = 2\pi\Big[2^3\Big(-\frac{1}{2}+1\Big) - \int_{1/2}^{1} u^{-3}du\Big] = 2\pi\Big[4 - \Big(\frac{u^{-2}}{-2}\Big|_{1/2}^{1}\Big)\Big],$$

$$I = 2\pi\Big[4 + \frac{1}{2}\big(u^{-2}\big|_{1/2}^{1}\big)\Big] = 2\pi\Big[4 + \frac{1}{2}\big(1-2^2\big)\Big] = 2\pi\Big[\frac{8}{2} - \frac{3}{2}\Big]$$

We conclude: $I = 5\pi$. ◁

Triple integral in spherical coordinates (Sect. 15.7)

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the half sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

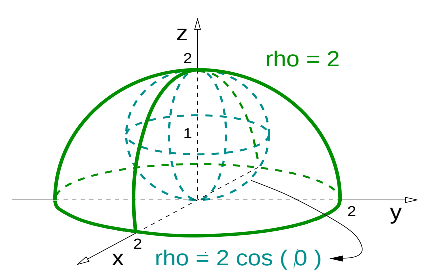
Solution: First sketch the integration region.

- $\rho = 2\cos(\phi)$ is a sphere, since

$$\rho^2 = 2\rho\cos(\phi) \Leftrightarrow x^2+y^2+z^2 = 2z$$

$$x^2 + y^2 + (z-1)^2 = 1.$$

- $\rho = 2$ is a sphere radius 2 and $\phi \in [0, \pi/2]$ says we only consider the upper half of the sphere.



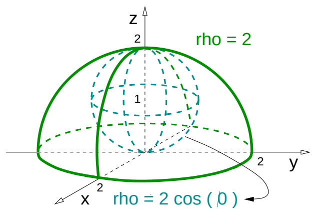
Triple integral in spherical coordinates (Sect. 15.7)

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution:

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2\cos(\phi)}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$



$$V = 2\pi \int_0^{\pi/2} \Big(\frac{\rho^3}{3} \Big|_{2\cos(\phi)}^{2} \Big) \sin(\phi)\, d\phi$$

$$V = \frac{2\pi}{3} \int_0^{\pi/2} \Big[8\sin(\phi) - 8\cos^3(\phi)\, \sin(\phi) \Big]\, d\phi.$$

$$V = \frac{16\pi}{3} \Big[\Big(-\cos(\phi) \Big|_0^{\pi/2} \Big) - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi \Big].$$

Triple integral in spherical coordinates (Sect. 15.7)

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2\cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution: $V = \frac{16\pi}{3} \Big[\Big(-\cos(\phi) \Big|_0^{\pi/2} \Big) - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi)\, d\phi \Big].$

Introduce the substitution: $u = \cos(\phi)$, $du = -\sin(\phi)\, d\phi$.

$$V = \frac{16\pi}{3} \Big[1 + \int_1^0 u^3\, du \Big] = \frac{16\pi}{3} \Big[1 + \Big(\frac{u^4}{4} \Big|_1^0 \Big) \Big] = \frac{16\pi}{3} \Big(1 - \frac{1}{4} \Big).$$

$$V = \frac{16\pi}{3} \frac{3}{4} \quad \Rightarrow \quad V = 4\pi. \qquad \triangleleft$$