Lines and planes in space (Sect. 12.5)

Planes in space.

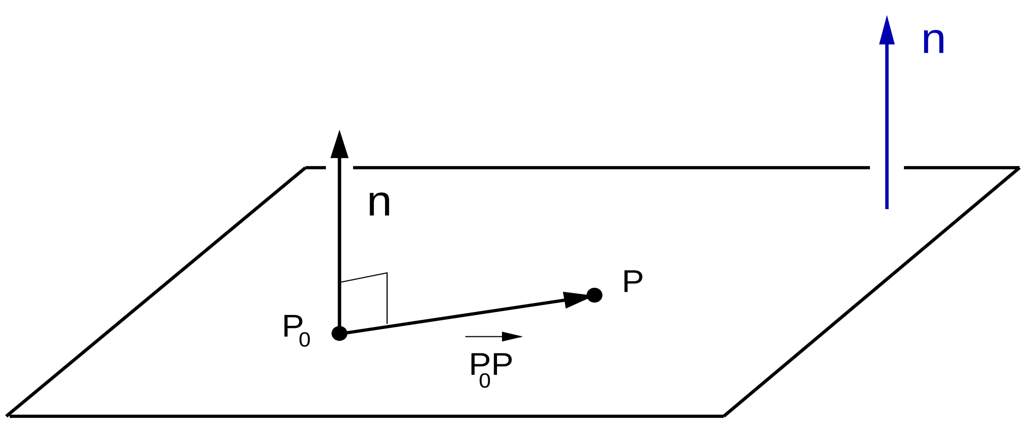
- Equations of planes in space.
  - Vector equation.
  - Components equation.
- The line of intersection of two planes.
- Parallel planes and angle between planes.
- Distance from a point to a plane.

A point an a vector determine a plane.

Definition

The *plane* by a point P_0 perpendicular to a non-zero vector $\mathbf{n}$, called the *normal vector*, is the set of points P solution of the equation

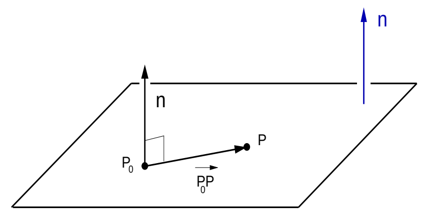
$$(\overrightarrow{P_0P}) \cdot \mathbf{n} = 0.$$



A point an a vector determine a plane.

Example

Does the point $P = (1, 2, 3)$ belong to the plane containing $P_0 = (3, 1, 2)$ and perpendicular to $\mathbf{n} = \langle 1, 1, 1\rangle$?



Solution: We need to know if the vector $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$. We first compute $\overrightarrow{P_0P}$,

$$\overrightarrow{P_0P} = \langle (1-3), (2-1), (3-2)\rangle \quad \Rightarrow \quad \overrightarrow{P_0P} = \langle -2, 1, 1\rangle.$$

This vector is orthogonal to $\mathbf{n}$, since

$$\big(\overrightarrow{P_0P}\big) \cdot \mathbf{n} = -2 + 1 + 1 = 0.$$

We conclude that P belongs to the plane. ◁

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Equation of a plane in Cartesian coordinates

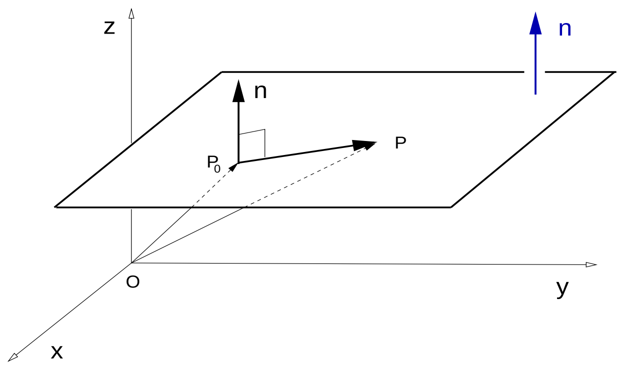
Theorem

Given any Cartesian coordinate system, the point $P = (x, y, z)$ *belongs to the plane by* $P_0 = (x_0, y_0, z_0)$ *perpendicular to* $\mathbf{n} = \langle n_x, n_y, n_z \rangle$ *iff holds*

$$(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0.$$

Furthermore, the equation above can be written as

$$n_x x + n_y y + n_z z = d, \qquad d = n_x x_0 + n_y y_0 + n_z z_0.$$



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Proof: In Cartesian coordinates

$$\overrightarrow{P_0P} = \langle (x - x_0), (y - y_0), (z - z_0) \rangle.$$

Therefore, the equation of the plane, $\overrightarrow{P_0P} \perp \mathbf{n}$, is

$$0 = \big(\overrightarrow{P_0P}\big) \cdot \mathbf{n} = (x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z. \qquad \square$$

Equation of a plane in Cartesian coordinates

Example

Find the equation of a plane containing $P_0 = (1, 2, 3)$ and perpendicular to $\mathbf{n} = \langle 1, -1, 2 \rangle$.

Solution: The point $P = (x, y, z)$ belongs to the plane above iff $(\overrightarrow{P_0P}) \cdot \mathbf{n} = 0$, that is,

$$\langle (x-1), (y-2), (z-3) \rangle \cdot \langle 1, -1, 2 \rangle = 0.$$

Computing the dot product above we get

$$(x-1) - (y-2) + 2(z-3) = 0.$$

The equation of the plane can be also written as

$$x - y + 2z = 5. \qquad \triangleleft$$

Equation of a plane in Cartesian coordinates

Example

Find a point P_0 and the perpendicular vector $\mathbf{n}$ to the plane $2x + 4y - z = 3$.

Solution: The equation of a plane is $n_x x + n_y y + n_z z = d$.

The components of the normal vector $\mathbf{n}$ are the coefficients that multiply the variables x, y and z. Hence,

$$\mathbf{n} = \langle 2, 4, -1 \rangle.$$

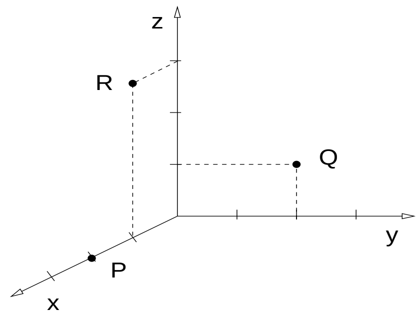
A point P_0 on the plane is simple to find. Just look for the intersection of the plane with one of the coordinate axis.

For example: set $y = 0$, $z = 0$ and find x from the equation of the plane: $2x = 3$, that is $x = 3/2$. Therefore, $P_0 = (3/2, 0, 0)$. $\triangleleft$

Equation of a plane in Cartesian coordinates

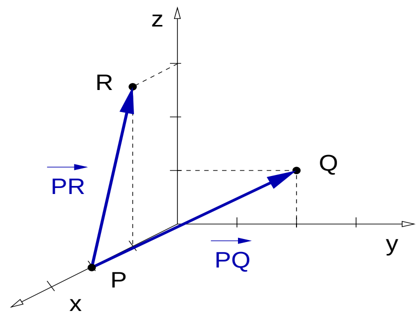
Example

Find the equation of the plane containing the points $P = (2,0,0)$, $Q = (0,2,1)$, $R = (1,0,3)$.



Solution:

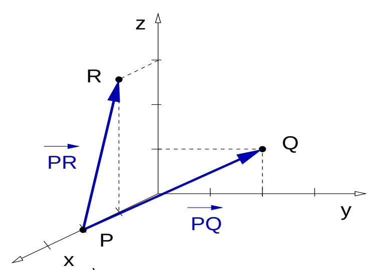
Find two tangent vectors to the plane, for example, $\overrightarrow{PQ} = \langle -2, 2, 1\rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3\rangle$.



Equation of a plane in Cartesian coordinates

Solution:

Find two tangent vectors to the plane, for example, $\overrightarrow{PQ} = \langle -2, 2, 1\rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3\rangle$.



Find a vector $\mathbf{n}$ perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
One way is using the cross product: $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$. That is,

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ -1 & 0 & 3 \end{vmatrix} = (6-0)\,\mathbf{i} - (-6+1)\,\mathbf{j} + (0+2)\,\mathbf{k}.$$

The result is: $\mathbf{n} = \langle 6, 5, 2\rangle$. Choose any point on the plane, say $P = (2,0,0)$. Then, the equation of the plane is:

$$6(x-2) + 5y + 2z = 0.$$

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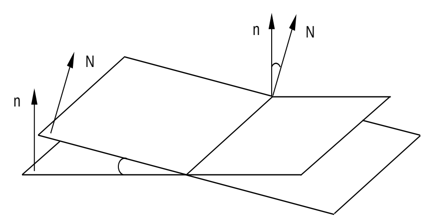
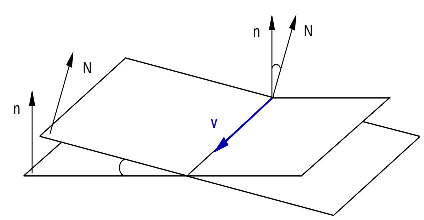
The line of intersection of two planes.

Example

Find a vector tangent to the line of intersection of the planes $2x + y - 3z = 2$ and $-x + 2y - z = 1$.



Solution:



We need to find a vector perpendicular to both normal vectors $\mathbf{n} = \langle 2, 1, -3\rangle$ and $\mathbf{N} = \langle -1, 2, -1\rangle$.

We choose $\mathbf{v} = \mathbf{N} \times \mathbf{n}$. That is,

$$\mathbf{v} = \mathbf{N} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -1 \\ 2 & 1 & -3 \end{vmatrix} = (-6+1)\,\mathbf{i} - (3+2)\,\mathbf{j} + (-1-4)\,\mathbf{k}$$

Result: $\mathbf{v} = \langle -5, -5, -5\rangle$. A simpler choice is $\mathbf{v} = \langle 1, 1, 1\rangle$. $\triangleleft$

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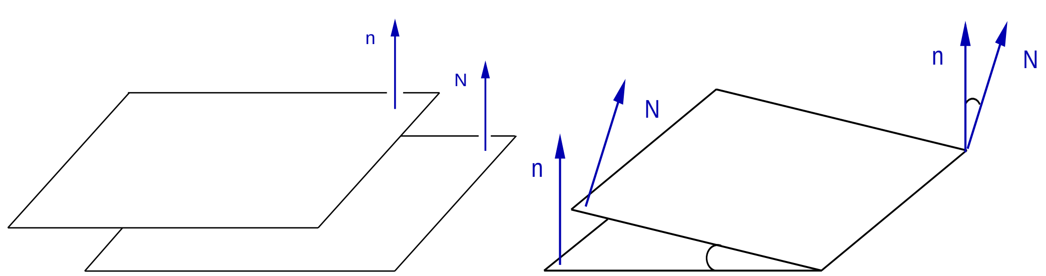
Parallel planes and angle between planes

Definition

Two planes are *parallel* if their normal vectors are parallel. The *angle* between two non-parallel planes is the smaller angle between their normal vectors.



Parallel planes and angle between planes

Example

Find the angle between the planes $2x + y - 3z = 2$ and $-x + 2y - z = 1$.

Solution: We need to find the angle between the normal vectors $\mathbf{n} = \langle 2, 1, -3 \rangle$ and $\mathbf{N} = \langle -1, 2, -1 \rangle$.

We use the dot product: $\cos(\theta) = \dfrac{\mathbf{n} \cdot \mathbf{N}}{|\mathbf{n}|\,|\mathbf{N}|}$.

The numbers we need are:

$$\mathbf{n} \cdot \mathbf{N} = -2 + 2 + 3 = 3,$$

$$|\mathbf{n}| = \sqrt{4 + 1 + 9} = \sqrt{14}, \qquad |\mathbf{N}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

Therefore, $\cos(\theta) = 3/\sqrt{84}$. We conclude that

$$\theta = 70^\circ\ 53'\ 36''.$$

$\triangleleft$

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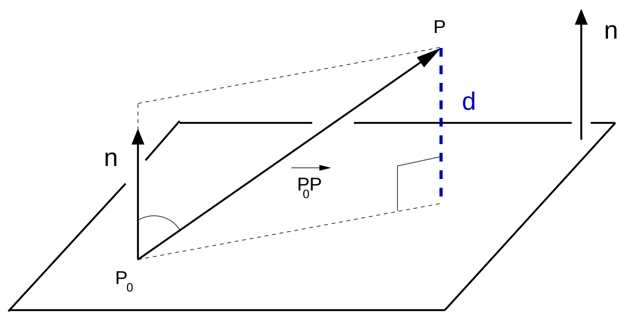
Distance formula from a point to a plane

Theorem

The distance d from a point P to a plane containing P_0 with normal vector **n** *is the shortest distance from P to any point in the plane, and is given by the expression*

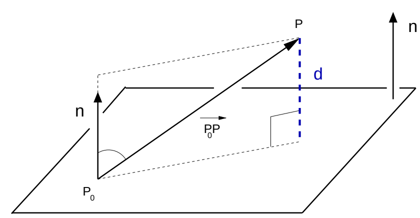
$$d = \frac{|(\overrightarrow{P_0P}) \cdot \mathbf{n}|}{|\mathbf{n}|}.$$



Distance formula from a point to a plane

Proof: It is simple to obtain the distance formula

$$d = \frac{|(\overrightarrow{P_0P}) \cdot \mathbf{n}|}{|\mathbf{n}|}.$$



From the picture above, and denoting θ is the angle between $\overrightarrow{P_0P}$ and **n**, we see that

$$d = \left||\overrightarrow{P_0P}|\cos(\theta)\right| = \left|\frac{(\overrightarrow{P_0P}) \cdot \mathbf{n}}{|\mathbf{n}|}\right|.$$

$\square$

Distance formula from a point to a plane

Example

Find the distance from the point $P = (1, 2, 3)$ to the plane $x - 3y + 2z = 4$.

Solution: We need to find a point P_0 on the plane and its normal vector $\mathbf{n}$. Then, use the formula $d = |(\overrightarrow{P_0P}) \cdot \mathbf{n}|/|\mathbf{n}|$.

To find a point on the plane: for example, if $y = 0$, $z = 0$, then $x = 4$. That is, $P_0 = (4, 0, 0)$.

The normal vector is in the plane equation: $\mathbf{n} = \langle 1, -3, 2 \rangle$.

We now compute $\overrightarrow{P_0P} = \langle -3, 2, 3 \rangle$. Then,

$$d = \frac{|-3 - 6 + 6|}{\sqrt{1 + 9 + 4}} \quad \Rightarrow \quad d = \frac{3}{\sqrt{14}}. \qquad \triangleleft$$