Name: ______________________________ **ID Number:** ____________________

TA: ______________________________ **Section Time:** __________________

MTH 234
Exam 2: Practice
October 19, 2010
50 minutes
Sects: 13.1, 13.3,
14.1-14.7.

No calculators or any other devices allowed.
If any question is not clear, ask for clarification.
No credit will be given for illegible solutions.
If you present different answers for the same problem, the worst answer will be graded.
Show all your work. ***Box your answers.***

1. (a) (15 points) Find the position $\boldsymbol{r}$ and velocity vector functions $\boldsymbol{v}$ of a particle that moves with an acceleration function $\boldsymbol{a}(t) = \langle 0, 0, -10 \rangle \; m/sec^2$, knowing that the initial velocity and position are given by, respectively, $\boldsymbol{v}(0) = \langle 0, 1, 2 \rangle \; m/sec$ and $\boldsymbol{r}(0) = \langle 0, 0, 3 \rangle \; m$.

(b) (5 points) Draw an approximate picture of the graph of $\boldsymbol{r}(t)$ for $t \geq 0$.

SOLUTION:

(a)

$$\boldsymbol{a}(t) = \langle 0, 0, -10 \rangle,$$

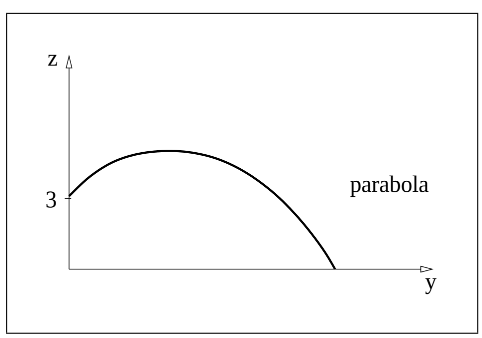
$$\boldsymbol{v}(t) = \langle v_{0x}, v_{0y}, -10\,t + v_{0z} \rangle, \quad \boldsymbol{v}(0) = \langle 0, 1, 2 \rangle \quad \Rightarrow \begin{cases} v_{0x} = 0, \\ v_{0y} = 1, \\ v_{0z} = 2. \end{cases}$$

$$\boldsymbol{v}(t) = \langle 0, 1, -10\,t + 2 \rangle.$$

$$\boldsymbol{r}(t) = \langle r_{0x}, t + r_{0y}, -5t^2 + 2t + r_{0z} \rangle, \quad \boldsymbol{r}(0) = \langle 0, 0, 3 \rangle \quad \Rightarrow \begin{cases} r_{0x} = 0, \\ r_{0y} = 0, \\ r_{0z} = 3. \end{cases}$$

$$\boxed{\boldsymbol{r}(t) = \langle 0, t, -5t^2 + 2t + 3 \rangle}.$$

(b)



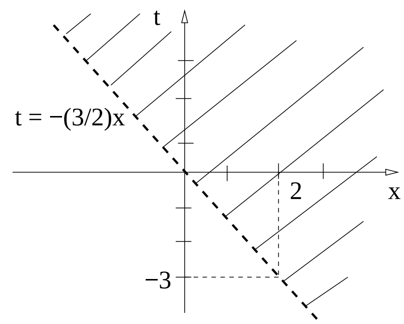
2. (a) (10 points) Find and sketch the domain of the function $f(x,t) = \ln(3x + 2t)$.

(b) (10 points) Find all possible constants c such that the function $f(x,t)$ above is solution of the wave equation, $f_{tt} - c^2 f_{xx} = 0$.

SOLUTION:

(a) The argument in the ln function must be positive. then, the domain is

$$\boxed{D = \{(x,t) \in \mathbb{R}^2 \ : \ 3x + 2t > 0\}}.$$



(b)

$$f_t = \frac{2}{3x + 2t}, \qquad f_x = \frac{3}{3x + 2t},$$

$$f_{tt} = -\frac{4}{(3x + 2t)^2}, \qquad f_{xx} = -\frac{9}{(3x + 2t)^2},$$

$$0 = f_{tt} - c f_{xx} = -\frac{4}{(3x + 2t)^2} + c^2 \frac{9}{(3x + 2t)^2} = \frac{1}{(3x + 2t)^2}(-4 + 9c^2) \quad \Rightarrow$$

$$\Rightarrow \quad 9c^2 = 4, \quad \Rightarrow \boxed{c = \pm\frac{2}{3}}.$$

3. (a) (10 points) Find the direction in which $f(x, y)$ increases the most rapidly, and the directions in which $f(x, y)$ decreases the most rapidly at P_0, and also find the value of the directional derivative of $f(x, y)$ at P_0 along these directions, where

$$f(x, y) = x^3 e^{-2y}, \quad \text{and} \quad P_0 = (1, 0).$$

(b) (10 points) Find the directional derivative of $f(x, y)$ above at the point P_0 in the direction given by $\boldsymbol{v} = \langle 1, -1 \rangle$.

SOLUTION:

(a) The direction in which f increases the most rapidly is given by ∇f, and the one in which decreases the most rapidly is $-\nabla f$. So,

$$\nabla f(x, y) = \langle 3x^2 e^{-2y}, -2x^3 e^{-2y} \rangle, \quad \Rightarrow \quad \boxed{\nabla f(1, 0) = \langle 3, -2 \rangle}, \quad \boxed{-\nabla f(1, 0) = \langle -3, 2 \rangle}.$$

The value of the directional derivative along these directions is, respectively, $|\nabla f(1, 0)|$ and $-|\nabla f(1, 0)|$, where

$$\boxed{|\nabla f(1, 0)| = \sqrt{9 + 4} = \sqrt{13}}.$$

(b) A unit vector along $\langle 1, -1 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$, then,

$$D_u f(1, 0) = \nabla f(1, 0) \cdot \mathbf{u} = \langle 3, -2 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, -1 \rangle = \frac{5}{\sqrt{2}},$$

$$\boxed{D_u f(1, 0) = \frac{5}{\sqrt{2}}}.$$

4. (a) (10 points) Find the tangent plane approximation of $f(x, y) = x\cos(\pi y/2) - y^2 e^{-x}$ at the point $(0, 1)$.

(b) (10 points) Use the linear approximation computed above to approximate the value of $f(-0.1, 0.9)$.

SOLUTION:

(a)

$$\begin{aligned} f(x,y) &= x\cos(\pi y/2) - y^2e^{-x} & f(0,1) &= -1, \\ f_x(x,y) &= \cos(\pi y/2) + y^2e^{-x} & f_x(0,1) &= \cos(\pi/2) + 1 = 1, \\ f_y(x,y) &= -x\sin(\pi y/2)\frac{\pi}{2} - 2ye^{-x} & f_y(0,1) &= -2, \end{aligned}$$

Then, the linear approximation $L(x, y)$ is given by

$$L(x,y) = (x-0) - 2(y-1) - 1, \quad \Rightarrow \quad \boxed{L(x,y) = x - 2y + 1}.$$

(b) The linear approximation of $f(-0.1, 0.9)$ is $L(-0.1, 0.9)$, which is given by

$$L(-0.1, 0.9) = -0.1 - 2(-0.1) - 1 = -0.1 - 1 = -1.1, \quad \Rightarrow \quad \boxed{L(-0.1, 0.9) = -1.1}.$$

5. (20 points) Find every local and absolute extrema of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \leq 1$, and indicate which ones are the absolute extrema. In the case of the interior stationary points, decide whether they are local maximum, minimum of saddle points.

SOLUTION:

We first compute the interior stationary points, which are (x, y) solutions of

$$\nabla f = \langle 2x, 6y + 2\rangle = \langle 0, 0\rangle \quad \Rightarrow \quad x = 0, \quad y = -\frac{1}{3}.$$

The point $(0, -1/3)$ belongs to the disk $x^2 + y^2 \leq 1$ so we have to decide whether it is a local maximum, minimum or saddle point:

$$f_{xx} = 2, \quad f_{yy} = 6, \quad f_{xy} = 0,$$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 12 > 0, \quad f_{xx} > 0 \quad \Rightarrow \quad \boxed{\Big(0, -\frac{1}{3}\Big) \text{ is a local minimum}}.$$

This point is also a candidate for absolute minimum, so we record the value of f,

$$\Big(0, -\frac{1}{3}\Big) \quad \Rightarrow \quad f\Big(0, -\frac{1}{3}\Big) = 0 + \frac{3}{9} - \frac{2}{3} = -\frac{1}{3}.$$

We now look for extreme point on the boundary $x^2 + y^2 = 1$. We evaluate $f(x, y)$ along the boundary. From the equation $x^2 + y^2 = 1$ we compute $x = \pm\sqrt{1 - y^2}$. This function is differentiable for $y \in (-1, 1)$, but is not differentiable at $y = \pm 1$. Since we need to use the chain rule to find the extrema of $g(y) = f(x(y), y)$ and the chain rule does not hold at $y = \pm 1$, we need to consider these points, $(0, \pm 1)$ separately:

$$(0, 1) \quad \Rightarrow \quad f(0, 1) = 5, \qquad (0, -1) \quad \Rightarrow \quad f(0, -1) = 1.$$

Now we find local extrema on $g(y) = f(x(y), y)$ in the interval $y \in (-1, 1)$. The function g is given by

$$g(y) = (1 - y^2) + 3y^2 + 2y \quad \Rightarrow \quad g(y) = 1 + 2y^2 + 2y.$$

The local extrema for g are the points y solutions of $g'(y) = 0$, that is, $4y + 2 = 0$, so we conclude $y = -1/2$ and $x = \pm\sqrt{1 - 1/4} = \pm\sqrt{3}/2$, that is,

$$\Big(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\Big) \quad \Rightarrow \quad f\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{3}{4} + \frac{3}{4} - 2\frac{1}{2} = \frac{1}{2}.$$

Therefore, the absolute extrema are

$$\boxed{(0, 1) \text{ absolute maximum}}, \qquad \boxed{\Big(0, -\frac{1}{3}\Big) \text{ absolute minimum}}.$$