Review for Exam 3

- Thursday Recitations: 15.1-15.5, 15.7.
- 50 minutes.
- From five 10-minute problems to ten 5-minutes problems.
- Problems similar to homework problems.
- No calculators, no notes, no books, no phones.

Double integrals in Cartesian coordinates (Section 15.2)

Example

Switch the integration order in
\[ I = \int_{0}^{3} \int_{-2}^{2} \frac{1 - \frac{2}{3}x}{\sqrt{1 - \frac{x^2}{3^2}}} f(x, y) \, dy \, dx. \]

Solution:
We first draw the integration region. Start with the outer limits.
\[ x \in [0, 3]. \]
\[ y \leq 2 - 2x/3 \text{ and } y \geq 2 \sqrt{1 - \frac{x^2}{3^2}}. \]

The lower limit is part of the ellipse
\[ \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1. \]
Double integrals in Cartesian coordinates (Section 15.2)

Example
Switch the integration order in

\[ I = \int_{0}^{3} \int_{-2}^{2} \frac{f(x, y)}{1 - \frac{x^2}{3^2}} \, dy \, dx. \]

Solution:

Split the integral at \( y = 0 \).

In \( y \in [-2, 0] \), holds \( 0 \leq x \).

The upper limit comes from

\[ \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1, \]

so, \( x = +3 \sqrt{1 - \frac{y^2}{2^2}} \).

In \( y \in [0, 2] \), holds \( 0 \leq x \). The upper limit comes from

\[ y = 2 \left(1 - \frac{x}{3}\right), \quad \text{that is,} \quad x = 3 \left(1 - \frac{y}{2}\right). \]

We then conclude:

\[ I = \int_{-2}^{2} \int_{0}^{3 \sqrt{1 - \frac{y^2}{2^2}}} f(x, y) \, dx \, dy + \int_{0}^{2} \int_{0}^{3 \left(1 - \frac{y}{2}\right)} f(x, y) \, dx \, dy. \]

\( \triangle \)

Areas as double integrals (Section 15.3)

Example
Compute the area of the region on the \( xy \)-plane below the curve \( y = 4 - x^2 \) and above \( y = x^2 \). Also switch the integration order.

Solution: First, sketch the integration region.

It is simpler integrating \( dy \, dx \).

\[ A = \int_{-2}^{2} \int_{x^2}^{4-x^2} dy \, dx. \]

\[ A = \int_{-2}^{2} [(4 - x^2) - x^2] \, dx \]

\[ A = \int_{-2}^{2} (4 - 2x^2) \, dx = 4x \bigg|_{-2}^{2} - \frac{2}{3} x^3 \bigg|_{-2}^{2} = (8 + 8) - \frac{2}{3} (8 + 8) = \frac{16}{3}. \]
Areas as double integrals (Section 15.3)

Example

Compute the area of the region on the xy-plane below the curve $y = 4 - x^2$ and above $y = x^2$. Also switch the integration order.

Solution: We now interchange the integration region to $dx \, dy$.

We need to divide the $y$-interval at $y$ such that

$$4 - x^2 = x^2 \Rightarrow x = \pm \sqrt{2}.$$ 

That is, $y = 2$. Then,

$$A = \int_{0}^{2} \int_{\sqrt{y}}^{-\sqrt{y}} dx \, dy + \int_{2}^{4} \int_{\sqrt{4-y}}^{-\sqrt{4-y}} dx \, dy.$$

Double integrals in polar coordinates. (Sect. 15.4)

Example

Transform to polar coordinates and then evaluate the integral

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx.$$

Solution: First sketch the integration region.

- $x \in [-2, \sqrt{2}]$.
- For $x \in [-2, -\sqrt{2}]$, we have $|y| \leq \sqrt{4 - x^2}$, so the curve is part of the circle $x^2 + y^2 = 4$.
- For $x \in [-\sqrt{2}, \sqrt{2}]$, we have that $y$ is between the line $y = x$ and the upper side of the circle $x^2 + y^2 = 4$. 

Example
Transform to polar coordinates and then evaluate the integral
\[ I = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx + \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx. \]

Solution:
\[ I = \int_{\pi/4}^{5\pi/4} \int_{0}^{r} r^2 \, rdr \, d\theta \]
\[ I = \left( \frac{5\pi}{4} - \frac{\pi}{4} \right) \int_{0}^{2} r^3 \, dr \]
\[ I = \pi \left( \frac{r^4}{4} \right) \bigg|_{0}^{2} \]
\[ I = 4\pi. \]

\[ x \in [-2, \sqrt{2}]. \]
\[ \text{For } x \in [-2, 0], \text{ we have } y \geq 0 \text{ and } y \leq \sqrt{4-x^2}. \text{ The latter curve is part of the circle } x^2 + y^2 = 4. \]
\[ \text{For } x \in [0, \sqrt{2}], \text{ we have } y \geq x \text{ and } y \leq \sqrt{4-x^2}. \]
**Double integrals in polar coordinates. (Sect. 15.4)**

**Example**
Transform to polar coordinates and then evaluate the integral

\[
I = \int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx + \int_{0}^{\sqrt{2}} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx
\]

**Solution:**

\[
I = \int_{\pi/4}^{\pi} \int_{0}^{2} r^2 \, rdr \, d\theta
\]

\[
I = \frac{3\pi}{4} \left( \frac{r^4}{4} \bigg|_0^2 \right)
\]

We conclude: \( I = 3\pi \).

**Triple integral in Cartesian coordinates (Sect. 15.5)**

**Example**

Find the volume of the region in the first octant below the plane \( 2x + y - 2z = 2 \) and \( x \leq 1, y \leq 2 \).

**Solution:** First sketch the integration region.

The plane contains the points \((1, 0, 0), (0, 2, 0), (1, 2, 1)\).

We choose the order \( dz \, dy \, dx \).

The integral is

\[
V = \int_{0}^{1} \int_{2-2x}^{2} \int_{0}^{-1+x+y/2} dz \, dy \, dx.
\]

\[
V = \int_{0}^{1} \int_{2-2x}^{2} \left[ (-1 + x) + \frac{y}{2} \right] dy \, dx,
\]

\[
V = \int_{0}^{1} \left[ -(1 - x)[2 - 2(1 - x)] + \frac{1}{4}[4 - 4(1 - x)^2] \right] dx.
\]
Example
Find the volume of the region in the first octant below the plane $2x + y - 2z = 2$ and $x \leq 1$, $y \leq 2$.

Solution: $V = \int_0^1 \int_0^{2(1-x)} \left[ - (1-x)[2-2(1-x)] + \frac{1}{4}[4-4(1-x)^2] \right] \, dx$.

$$V = \int_0^1 \left[ - 2(1-x) + 2(1-x)^2 + 1 - (1-x)^2 \right] \, dx,$$

$$V = \int_0^1 \left[ - 1 + 2x + (1-x)^2 \right] \, dx = \int_0^1 \left[ - 1 + 2x + 1 + x^2 - 2x \right] \, dx$$

$$V = \int_0^1 x^2 \, dx = \frac{x^3}{3} \Big|_0^1 \Rightarrow V = \frac{1}{3}.$$

Example
Find the volume of the region in the first octant below the plane $x + y + z = 3$ and $y \leq 1$.

Solution: First sketch the integration region.

The plane contains the points $(1, 0, 0)$, $(0, 2, 0)$, $(1, 2, 1)$.

We choose the order $dz \, dy \, dx$.
We need $x + y = 3$ at $z = 0$.

$$V = \int_0^2 \int_0^1 \int_0^{3-x-y} dz \, dy \, dx + \int_2^3 \int_0^1 \int_0^{3-x-y} dz \, dy \, dx.$$

$$V = \int_0^2 \int_0^1 (3 - x - y) \, dy \, dx + \int_2^3 \int_0^1 (3 - x - y) \, dy \, dx.$$
Example

Find the volume of the region in the first octant below the plane $x + y + z = 3$ and $y \leq 1$.

Solution:

$$V = \int_0^2 \int_0^1 (3 - x - y) \, dy \, dx + \int_2^3 \int_0^{3-x} (3 - x - y) \, dy \, dx.$$

$$V = \int_0^2 \left[ (3-x) \left( \int_0^1 \right) - \left( \frac{y^2}{2} \right)_0^1 + (3-x) \left( \int_0^{(3-x)} \right) - \left( \frac{y^2}{2} \right)_0^{(3-x)} \right] \, dx$$

$$V = \int_0^2 \left[ (3-x) - \frac{1}{2} + (3-x)^2 - \frac{1}{2}(3-x)^2 \right] \, dx$$

$$V = \int_0^2 \left[ \frac{5}{2} - x + \frac{1}{2}(3-x)^2 \right] \, dx \quad \Rightarrow \quad V = \frac{22}{3}. \quad \triangle$$

Example

Use spherical coordinates to find the volume of the region below the paraboloid $z = 9 - x^2 - y^2$ below the $xy$-plane and outside the cylinder $x^2 + y^2 = 1$.

Solution: First sketch the integration region.

In cylindrical coordinates,

$$z = 9 - x^2 - y^2 \iff z = 9 - r^2.$$ 

$$x^2 + y^2 = 1 \iff r = 1.$$

$$V = \int_0^{2\pi} \int_1^3 \int_0^{9-r^2} r \, dz \, dr \, d\theta = 2\pi \int_1^3 (9 - r^2) \, r \, dr$$

$$V = 2\pi \left( \frac{9r^2}{2} - \frac{r^4}{4} \right)_1^3 \quad \Rightarrow \quad V = 32\pi. \quad \triangle$$
Example
Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2 \cos(\phi)$ and inside the half sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution: First sketch the integration region.

- $\rho = 2 \cos(\phi)$ is a sphere, since
  \[
  \rho^2 = 2 \rho \cos(\phi) \iff x^2 + y^2 + z^2 = 2z \\
  x^2 + y^2 + (z - 1)^2 = 1.
  \]

- $\rho = 2$ is a sphere radius 2 and $\phi \in [0, \pi/2]$ says we only consider the upper half of the sphere.

\[
V = \int_0^{2\pi} \int_0^{\pi/2} \int_{2 \cos(\phi)}^{2} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta.
\]

\[
V = 2\pi \int_0^{\pi/2} \left( \frac{\rho^3}{3} \right)_{2 \cos(\phi)}^{2} \sin(\phi) \, d\phi
\]

\[
V = \frac{2\pi}{3} \int_0^{\pi/2} \left[ 8 \sin(\phi) - 8 \cos^3(\phi) \sin(\phi) \right] \, d\phi.
\]

\[
V = \frac{16\pi}{3} \left[ \left( - \cos(\phi) \right)_{0}^{\pi/2} \right] - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi) \, d\phi.
\]
Triple integral in spherical coordinates (Sect. 15.7)

Example

Use spherical coordinates to find the volume of the region outside the sphere $\rho = 2 \cos(\phi)$ and inside the sphere $\rho = 2$ with $\phi \in [0, \pi/2]$.

Solution: $V = \frac{16\pi}{3} \left[ -\cos(\phi) \big|_0^{\pi/2} \right] - \int_0^{\pi/2} \cos^3(\phi) \sin(\phi) \, d\phi$.

Introduce the substitution: $u = \cos(\phi)$, $du = -\sin(\phi) \, d\phi$.

$$V = \frac{16\pi}{3} \left[ 1 + \int_1^0 u^3 \, du \right] = \frac{16\pi}{3} \left[ 1 + \left( \frac{u^4}{4} \big|_0^1 \right) \right] = \frac{16\pi}{3} \left( 1 - \frac{1}{4} \right).$$

$$V = \frac{16\pi}{3} \cdot \frac{3}{4} \Rightarrow V = 4\pi. \quad \triangleright$$

Triple integral in cylindrical coordinates (Sect. 15.7)

Example

Use cylindrical coordinates to find the volume in the $z \geq 0$ region of a curved wedge cut out from a cylinder $(x - 2)^2 + y^2 = 4$ by the planes $z = 0$ and $z = -y$.

Solution: First sketch the integration region.

- $(x - 2)^2 + y^2 = 4$ is a circle in the xy-plane, since

  $$x^2 + y^2 = 4x \iff r^2 = 4r \cos(\theta) \Rightarrow r = 4 \cos(\theta).$$

- Since $0 \leq z \leq -y$, the integration region is on the $y \leq 0$ part of the $z = 0$ plane.
Triple integral in cylindrical coordinates (Sect. 15.7)

Example
Use cylindrical coordinates to find the volume in the \( z \geq 0 \) region of a curved wedge cut out from a cylinder \((x - 2)^2 + y^2 = 4\) by the planes \( z = 0 \) and \( z = -y \).

Solution:

\[
V = \int_{\frac{\pi}{2}}^{2\pi} \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{4\cos(\theta)} -r \sin(\theta) \, r \, dz \, dr \, d\theta.
\]

\[
V = \left[ -\frac{r^3}{3} \right]_{0}^{4\cos(\theta)} \sin(\theta) \, d\theta.
\]

\[
V = -\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{4^3}{3} \cos^3(\theta) \sin(\theta) \, d\theta.
\]

We conclude: \( V = \frac{16}{3} \).

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