Lines and planes in space (Sect. 12.5)

Lines in space (Today).

- Review: Lines on a plane.
- The equations of lines in space:
  - Vector equation.
  - Parametric equation.
- Distance from a point to a line.

Planes in space (Next class).

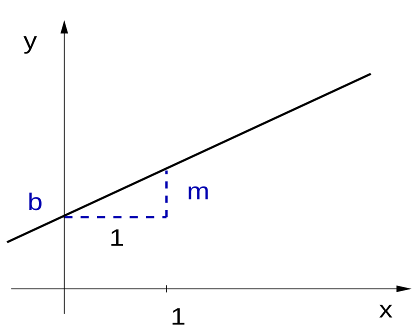
- Equations of planes in space.
  - Vector equation.
  - Components equation.
- The line of intersection of two planes.
- Parallel planes and angle between planes.
- Distance from a point to a plane.

Review: Lines on a plane

Equation of a line

The equation of a line with slope m and vertical intercept b is given by

$$y = mx + b.$$

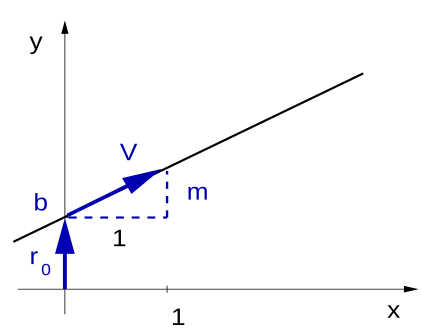


Vector equation of a line

The equation of the line by the point $P = (0, b)$ parallel to the vector $\mathbf{v} = \langle 1, m \rangle$ is given by

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v},$$

where $\mathbf{r}_0 = \overrightarrow{OP} = \langle 0, b \rangle$.



Review: Lines on a plane

Example

Find the vector equation of a line $y = -x + 3$.

Solution: The vertical intercept is at the point $P = (0, 3)$.

A vector tangent to the line is $\mathbf{v} = \langle 1, -1 \rangle$, since the point $P_1 = (1, 2)$ belongs to the line, which implies that $\mathbf{v} = \overrightarrow{PP_1} = \langle (1-0), (2-3) \rangle = \langle 1, -1 \rangle$.

The vector equation for the line is

$$\mathbf{r}(t) = \langle 0, 3 \rangle + t \langle 1, -1 \rangle.$$

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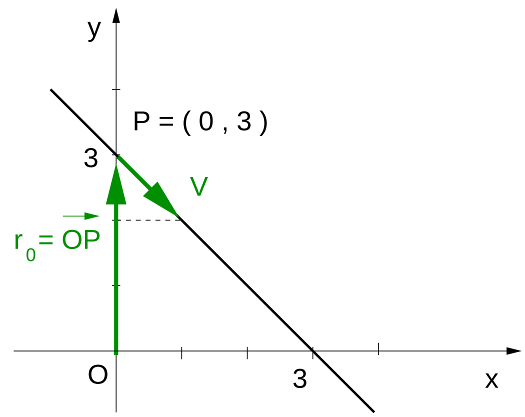

Review: Lines on a plane

Example

We verify the result above: That the line $y = -x + 3$ is indeed

$$\mathbf{r}(t) = \langle 0, 3 \rangle + t \langle 1, -1 \rangle, \quad \text{(Vector equation of the line.)}$$

Solution:

If $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then $\langle x(t), y(t) \rangle = \langle (0 + t), (3 - t) \rangle$. So,

$$x(t) = t, \qquad \text{(Parametric equation of the line.)}$$

$$y(t) = 3 - t. \qquad \text{(The parameter is } t.\text{)}$$

Replace t by x is the second equation,

$$y(x) = -x + 3.$$

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Review: Lines on a plane

Vector equation of a line

The equation of the line by the point $P = (0, b)$ parallel to the vector $\mathbf{v} = \langle 1, m\rangle$ is given by

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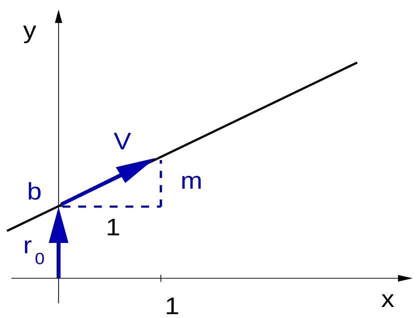

Parametric equation of a line

A line with vector equation

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v},$$

where $\mathbf{r}_0 = \langle 0, b\rangle$ and $\mathbf{v} = \langle 1, m\rangle$, can be written as follows,

$$\mathbf{r}(t) = \langle 0, b\rangle + t\,\langle 1, m\rangle,$$

$$\langle x(t), y(t)\rangle = \langle (0 + t), (b + tm)\rangle,$$

hence,

$$x(t) = t$$
$$y(t) = b + mt.$$

Lines and planes in space (Sect. 12.5)

Lines in space

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- **The equations of lines in space:**
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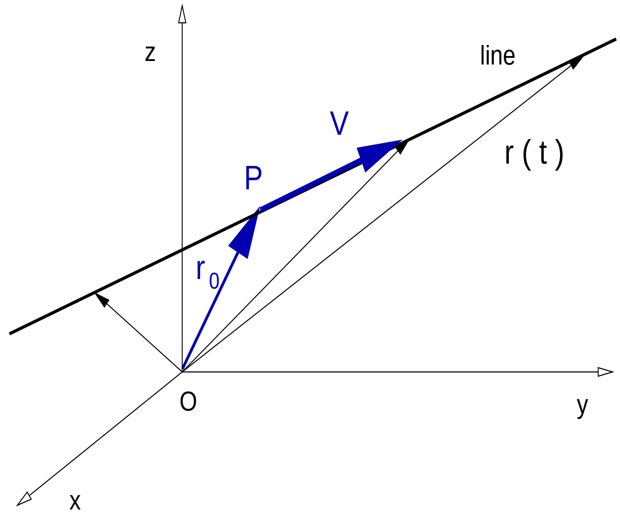
Vector equation of a line in space

Definition

The *vector equation of the line* by the point P parallel to the vector $\mathbf{v}$ is the set of terminal points of the vectors

$$\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}, \quad t \in \mathbb{R},$$

where $\mathbf{r}_0 = \overrightarrow{OP}$ and O is the origin of Cartesian coordinates in $\mathbb{R}^3$.



Remark: A line can refer to both the set of vectors $\mathbf{r}(t)$ and the set of terminal points of these vectors.

Vector equation of a line in space

Example

Find the vector equation of the line by the point $P = (1, -2, 1)$ tangent to the vector $\mathbf{v} = \langle 1, 2, 3 \rangle$.

Solution:
First, we construct the vector $\mathbf{r}_0 = \overrightarrow{OP} = \langle 1, -2, 1 \rangle$.
Then, the formula $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$ implies

$$\mathbf{r}(t) = \langle 1, -2, 1 \rangle + t\,\langle 1, 2, 3 \rangle.$$

$\triangleleft$

Lines and planes in space (Sect. 12.5)

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Parametric equation of a line in space

Definition

The *parametric equations of a line* by $P = (x_0, y_0, z_0)$ tangent to $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ are given by

$$x(t) = x_0 + t\, v_x,$$
$$y(t) = y_0 + t\, v_y,$$
$$z(t) = z_0 + t\, v_z.$$

Remark: It is simple to obtain the parametric equations form the vector equation, and vice-versa, noticing the relation

$$\mathbf{r}(t) = \mathbf{r}_0 + t\, \mathbf{v}$$
$$\langle x(t), y(t), z(t) \rangle = \langle x_0, y_0, z_0 \rangle + t\, \langle v_x, v_y, v_z \rangle$$
$$\langle x(t), y(t), z(t) \rangle = \langle (x_0 + t\, v_x), (y_0 + t\, v_y), (z_0 + t\, v_z) \rangle.$$

Parametric equation of a line in space

Example

Find the parametric equations of the line with vector equation

$$\mathbf{r}(t) = \langle 1, -2, 1 \rangle + t \langle 1, 2, 3 \rangle.$$

Solution: Rewrite the vector equation in vector components,

$$\langle x(t), y(t), z(t) \rangle = \langle (1 + t), (-2 + 2t), (1 + 3t) \rangle.$$

We conclude that

$$x(t) = 1 + t,$$
$$y(t) = -2 + 2t,$$
$$z(t) = 1 + 3t.$$

$\triangleleft$

Parametric equation of a line in space

Example

Find both the vector equation and the parametric equation of the line containing the points $P = (1, 2, -3)$ and $Q = (3, -2, 1)$.

Solution: A vector tangent to the line is $\mathbf{v} = \overrightarrow{PQ}$, given by

$$\mathbf{v} = \langle (3 - 1), (-2 - 2), (1 + 3) \rangle \quad \Rightarrow \quad \mathbf{v} = \langle 2, -4, 4 \rangle.$$

Either P or Q can be used in the vector equation of the line.

If we choose P, then

$$\mathbf{r}(t) = \langle 1, 2, -3 \rangle + t \langle 2, -4, 4 \rangle.$$

If we choose Q, then

$$\tilde{\mathbf{r}}(s) = \langle 3, -2, 1 \rangle + s \langle 2, -4, 4 \rangle.$$

Remark: t and s are different; $t = s + 1$, and $\mathbf{r}(s + 1) = \tilde{\mathbf{r}}(s)$.

Parametric equation of a line in space

Example

Find both the vector equation and the parametric equation of the line containing the points $P = (1, 2, -3)$ and $Q = (3, -2, 1)$.

Solution: The parametric equation of the line is simple to obtain once the vector equation is known. Since

$$\mathbf{r}(t) = \langle 1, 2, -3 \rangle + t \, \langle 2, -4, 4 \rangle,$$

$$\langle x(t), y(t), z(t) \rangle = \langle (1 + 2t), (2 - 4t), (-3 + 4t) \rangle.$$

Then, the parametric equations of the line are given by

$$x(t) = 1 + 2t,$$
$$y(t) = 2 - 4t,$$
$$z(t) = -3 + 4t.$$

$\lhd$

Lines and planes in space (Sect. 12.5)

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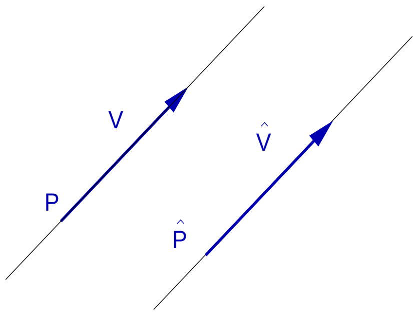
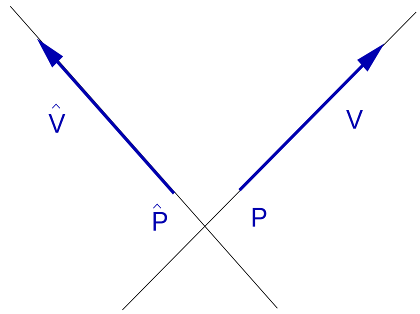
Parallel lines, perpendicular lines, intersections

Definition

The lines $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$ and $\hat{\mathbf{r}}(t) = \hat{\mathbf{r}}_0 + t\,\hat{\mathbf{v}}$ are *parallel* iff their tangent vectors $\mathbf{v}$ and $\hat{\mathbf{v}}$ are parallel; they are *perpendicular* iff $\mathbf{v}$ and $\hat{\mathbf{v}}$ are perpendicular; and the lines *intersect* iff they have a common point.



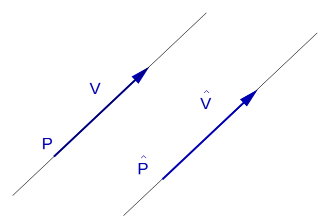


Remark: Perpendicular lines in space may not intersect.
Non-parallel lines in space may not intersect.

Parallel lines, perpendicular lines, intersections

Example

Find the line through $P = (1,1,1)$ and parallel to the line $\hat{\mathbf{r}}(t) = \langle 1,2,3\rangle + t\,\langle 2,-1,1\rangle$



Solution:
We need to find $\mathbf{r}_0$ and $\mathbf{v}$ such that $\mathbf{r}(t) = \mathbf{r}_0 + t\,\mathbf{v}$.

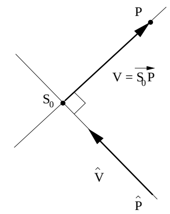
The vector $\mathbf{r}_0$ is simple to find: $\mathbf{r}_0 = \overrightarrow{OP} = \langle 1,1,1\rangle$.

The vector $\mathbf{v}$ is simple to find too: $\mathbf{v} = \langle 2,-1,1\rangle$.

We conclude: $\mathbf{r}(t) = \langle 1,1,1\rangle + t\,\langle 2,-1,1\rangle$. $\lhd$

Example

Find the line through $P = (1, 1, 1)$ perpendicular to and intersecting the line $\hat{\mathbf{r}}(t) = \langle 1, 2, 3\rangle + t\,\langle 2, -1, 1\rangle$



Solution:

Find a point S on the intersection such that $\overrightarrow{PS}$ is perpendicular to $\hat{\mathbf{v}} = \langle 2, -1, 1\rangle$. Writing $S_t = \hat{\mathbf{r}}(t) = \langle (1 + 2t), (2 - t), (3 + t)\rangle$,

$$\overrightarrow{PS_t} = \langle 2t, (1 - t), (2 + t)\rangle \perp \hat{\mathbf{v}} = \langle 2, -1, 1\rangle \quad \Leftrightarrow \quad \overrightarrow{PS_t} \cdot \hat{\mathbf{v}} = 0.$$

$$0 = \overrightarrow{PS_t} \cdot \hat{\mathbf{v}} = 4t - (1 - t) + (2 + t) = 6t + 1 \quad \Rightarrow \quad t_0 = -\frac{1}{6}.$$

$$\overrightarrow{PS_0} = \Big\langle -\frac{2}{6}, \Big(1 + \frac{1}{6}\Big), \Big(2 - \frac{1}{6}\Big)\Big\rangle \quad \Rightarrow \quad \overrightarrow{PS_0} = \frac{1}{6}\langle -2, 7, 11\rangle.$$

$$\mathbf{r}(t) = \overrightarrow{OP} + t\,\overrightarrow{PS_0} \quad \Rightarrow \quad \mathbf{r}(t) = \langle 1, 1, 1\rangle + \frac{t}{6}\,\langle -2, 7, 11\rangle. \qquad \triangleleft$$

Verify: $\langle -2, 7, 11\rangle \perp \langle 2, -1, 1\rangle$ and $\mathbf{r}(1) = \hat{\mathbf{r}}(-1/6)$.

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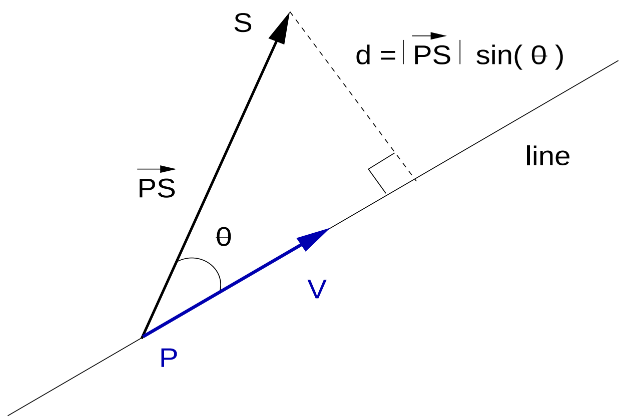
Distance from a point to a line in space

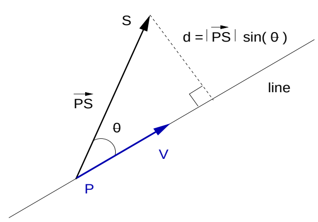
Theorem

The distance from a point S in space to a line through the point P with tangent vector $\mathbf{v}$ is given by

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$



Distance from a point to a line in space



Proof.

The distance from the point S to the line passing by the point P with tangent vector $\mathbf{v}$ is given by

$$d = |\overrightarrow{PS}| \sin(\theta).$$

Recalling that $|\overrightarrow{PS} \times \mathbf{v}| = |\overrightarrow{PS}|\,|\mathbf{v}| \sin(\theta)$, we conclude that

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}.$$

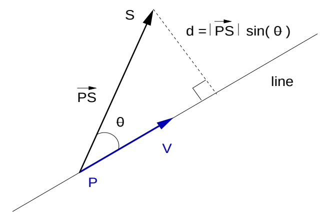
□

Distance from a point to a line in space

Example

Find the distance from the point
$S = (1, 2, 1)$ to the line

$x = 2 - t, \quad y = -1 + 2t, \quad z = 2 + 2t.$



Solution:
First we need to compute the vector equation of the line above.

The vector components are the numbers that multiply t.
This line has tangent vector $\mathbf{v} = \langle -1, 2, 2 \rangle$.

To find a point in the line, just evaluate it at $t = 0$.
This line contains the vector $P = (2, -1, 2)$.

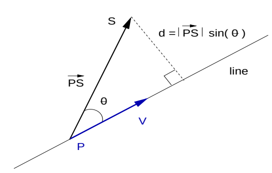
Therefore, $\overrightarrow{PS} = \langle -1, 3, -1 \rangle$.

Distance from a point to a line in space

Example

Find the distance from the point
$S = (1, 2, 1)$ to the line

$x = 2 - t, \quad y = -1 + 2t, \quad z = 2 + 2t.$



Solution: $P = (2, -1, 2)$, $\mathbf{v} = \langle -1, 2, 2 \rangle$, and $\overrightarrow{PS} = \langle -1, 3, -1 \rangle$.
Since $d = |\overrightarrow{PS} \times \mathbf{v}| / |\mathbf{v}|$, we need to compute:

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -1 \\ -1 & 2 & 2 \end{vmatrix} = (6 + 2)\,\mathbf{i} - (-2 - 1)\,\mathbf{j} + (-2 + 3)\,\mathbf{k},$$

that is, $\overrightarrow{PS} \times \mathbf{v} = \langle 8, 3, 1 \rangle$. We then compute the lengths:

$$|\overrightarrow{PS} \times \mathbf{v}| = \sqrt{64 + 9 + 1} = \sqrt{74}, \quad |\mathbf{v}| = \sqrt{1 + 4 + 4} = 3.$$

The distance from S to the line is $d = \sqrt{74}/3$. $\triangleleft$

Lines in space

Exercise

Consider the lines

$$x(t) = 1 + t, \qquad x(s) = 2s,$$
$$y(t) = \frac{3}{2} + 3t, \qquad y(s) = 1 + s,$$
$$z(t) = -t, \qquad z(s) = -2 + 4s.$$

Are the lines parallel? Do they intersect?

Answer:

The lines are not parallel.

The lines intersect at $P = \left(1, \frac{3}{2}, 0\right)$. ◁