Dot product and vector projections (Sect. 12.3)

- Two definitions for the dot product.
- Geometric definition of dot product.
- Orthogonal vectors.
- Dot product and orthogonal projections.
- Properties of the dot product.
- Dot product in vector components.
- Scalar and vector projection formulas.

Two main ways to introduce the dot product

Geometrical definition $\rightarrow$ Properties $\rightarrow$ Expression in components.

Definition in components $\rightarrow$ Properties $\rightarrow$ Geometrical expression.

We choose the first way, the textbook chooses the second way.

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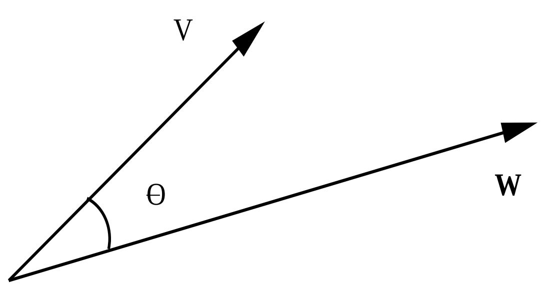
The dot product of two vectors is a scalar

Definition

The *dot product* of the vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, with $n = 2, 3$, having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \cdot \mathbf{w}$ and given by

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \, |\mathbf{w}| \cos(\theta).$$



Initial points together.

The dot product of two vectors is a scalar

Example

Compute $\mathbf{v} \cdot \mathbf{w}$ knowing that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, with $|\mathbf{v}| = 2$, $\mathbf{w} = \langle 1, 2, 3 \rangle$ and the angle in between is $\theta = \pi/4$.

Solution: We first compute $|\mathbf{w}|$, that is,

$$|\mathbf{w}|^2 = 1^2 + 2^2 + 3^2 = 14 \quad \Rightarrow \quad |\mathbf{w}| = \sqrt{14}.$$

We now use the definition of dot product:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = (2)\,\sqrt{14}\,\frac{\sqrt{2}}{2} \quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{w} = 2\sqrt{7}.$$

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- The angle between two vectors usually is not know in applications.
- It is useful to have a formula for the dot product involving the vector components.

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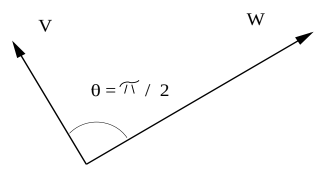
Perpendicular vectors have zero dot product.

Definition

Two vectors are *perpendicular*, also called *orthogonal*, iff the angle in between is $\theta = \pi/2$.



Theorem

The non-zero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular iff* $\mathbf{v} \cdot \mathbf{w} = 0$.

Proof.

$$\left.\begin{array}{c} 0 = \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \\ |\mathbf{v}| \neq 0, \quad |\mathbf{w}| \neq 0 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} \cos(\theta) = 0 \\ 0 \leqslant \theta \leqslant \pi \end{array}\right. \quad \Leftrightarrow \quad \theta = \frac{\pi}{2}.$$

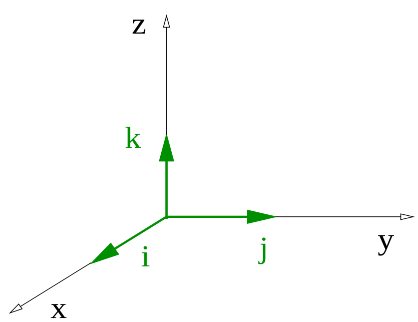
□

The dot product of i, j and k is simple to compute

Example

Compute all dot products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



$$\begin{array}{ccc} \mathbf{i} \cdot \mathbf{i} = 1, & \mathbf{j} \cdot \mathbf{j} = 1, & \mathbf{k} \cdot \mathbf{k} = 1, \\ \mathbf{i} \cdot \mathbf{j} = 0, & \mathbf{j} \cdot \mathbf{i} = 0, & \mathbf{k} \cdot \mathbf{i} = 0, \\ \mathbf{i} \cdot \mathbf{k} = 0, & \mathbf{j} \cdot \mathbf{k} = 0, & \mathbf{k} \cdot \mathbf{j} = 0. \end{array}$$

◁

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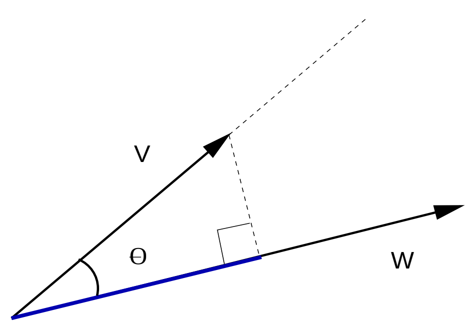
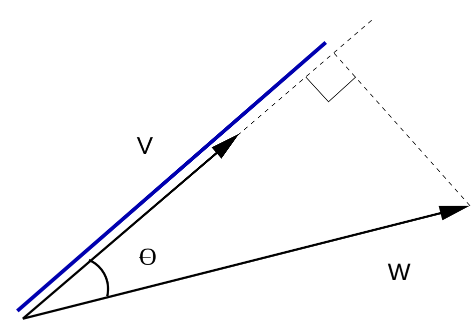
The dot product and orthogonal projections.

Remark: The dot product is closely related to orthogonal projections of one vector onto the other.

Recall: $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.





$$|\mathbf{v}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}. \qquad |\mathbf{w}|\,\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|}.$$

Remark: If $|\mathbf{u}| = 1$, then $\mathbf{v} \cdot \mathbf{u}$ is the projection of $\mathbf{v}$ along $\mathbf{u}$.

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Properties of the dot product.

Theorem

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, *(symmetric)*;

(b) $\mathbf{v} \cdot (a\mathbf{w}) = a\,(\mathbf{v} \cdot \mathbf{w})$, *(linear)*;

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(linear)*;

(d) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geqslant 0$, *and* $\mathbf{v} \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{0}$, *(positive)*;

(e) $\mathbf{0} \cdot \mathbf{v} = 0$.

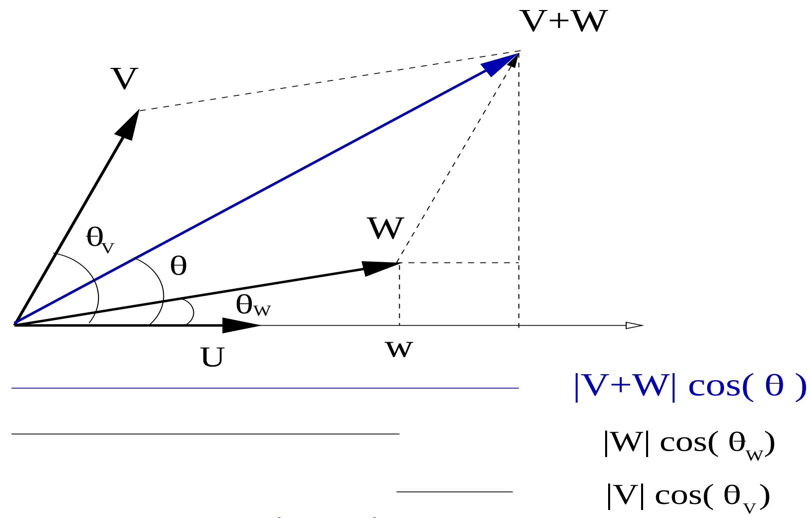
Proof.
Properties (a), (b), (d), (e) are simple to obtain from the definition of dot product $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.
For example, the proof of (b) for $a > 0$:

$$\mathbf{v} \cdot (a\mathbf{w}) = |\mathbf{v}|\,|a\mathbf{w}|\,\cos(\theta) = a\,|\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) = a\,(\mathbf{v} \cdot \mathbf{w}).$$

□

Properties of the dot product.

(c) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, is non-trivial. The proof is:



$$\left.\begin{aligned} |\mathbf{v} + \mathbf{w}| \cos(\theta) &= \frac{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})}{|\mathbf{u}|}, \\ |\mathbf{w}| \cos(\theta_w) &= \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}|}, \\ |\mathbf{v}| \cos(\theta_v) &= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}, \end{aligned}\right\} \Rightarrow \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

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The dot product in vector components (Case $\mathbb{R}^2$)

Theorem

If $\mathbf{v} = \langle v_x, v_y \rangle$ *and* $\mathbf{w} = \langle w_x, w_y \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

Proof.

Recall: $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$ and $\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j}$. The linear property of the dot product implies

$$\mathbf{v} \cdot \mathbf{w} = (v_x \mathbf{i} + v_y \mathbf{j}) \cdot (w_x \mathbf{i} + w_y \mathbf{j})$$

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x \mathbf{i} \cdot \mathbf{i} + v_x w_y \mathbf{i} \cdot \mathbf{j} + v_y w_x \mathbf{j} \cdot \mathbf{i} + v_y w_y \mathbf{j} \cdot \mathbf{j}.$$

Recall: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$. We conclude that

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y.$$

□

The dot product in vector components (Case $\mathbb{R}^3$)

Theorem

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ *and* $\mathbf{w} = \langle w_x, w_y, w_z \rangle$, *then* $\mathbf{v} \cdot \mathbf{w}$ *is given by*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

- The proof is similar to the case in $\mathbb{R}^2$.
- The dot product is simple to compute from the vector component formula $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$.
- The geometrical meaning of the dot product is simple to see from the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\, |\mathbf{w}| \cos(\theta)$.

Example

Find the cosine of the angle between $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = \langle 2, 1 \rangle$

Solution:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta) \quad \Rightarrow \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Furthermore,

$$\left.\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= (1)(2) + (2)(1) \\ |\mathbf{v}| &= \sqrt{1^2 + 2^2} = \sqrt{5}, \\ |\mathbf{w}| &= \sqrt{2^2 + 1^2} = \sqrt{5}, \end{aligned}\right\} \quad \Rightarrow \quad \cos(\theta) = \frac{4}{5}.$$

$\triangleleft$

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Scalar and vector projection formulas.

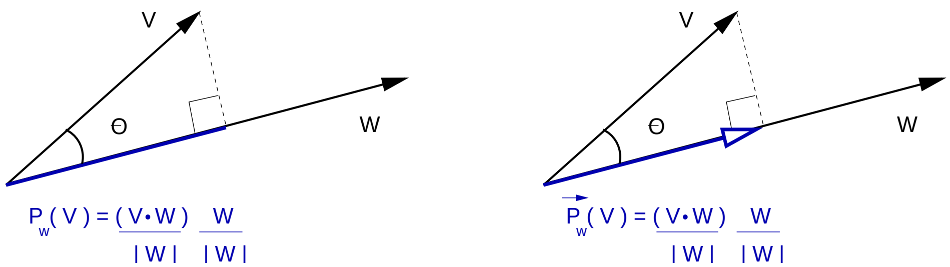
Theorem

The scalar projection of **v** *along* **w** *is the number* $p_w(v)$,

$$p_w(v) = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}.$$

The vector projection of **v** *along* **w** *is the vector* $\mathbf{p}_w(v)$,

$$\mathbf{p}_w(v) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|}\right) \frac{\mathbf{w}}{|\mathbf{w}|}.$$



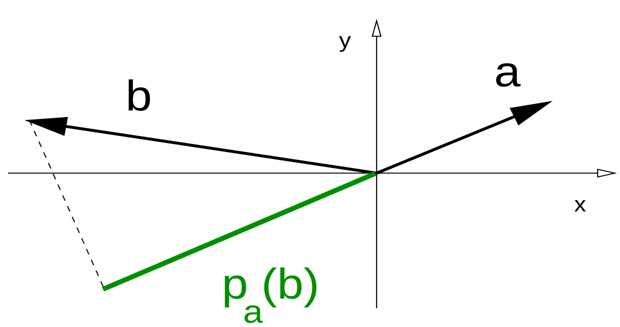
Example

Find the scalar projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The scalar projection of **b** onto **a** is the number

$$p_a(b) = |\mathbf{b}| \cos(\theta) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{(-4)(1) + (1)(2)}{\sqrt{1^2 + 2^2}}.$$

We therefore obtain $p_a(b) = -\frac{2}{\sqrt{5}}$.



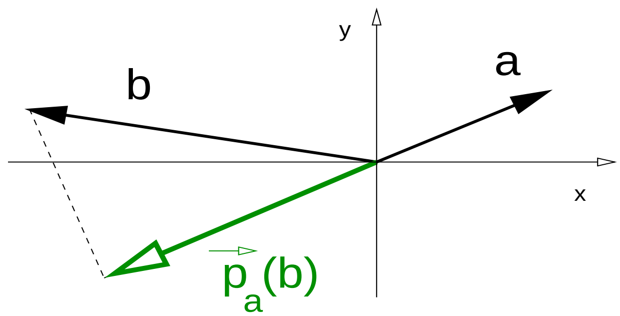
Example

Find the vector projection of $\mathbf{b} = \langle -4, 1 \rangle$ onto $\mathbf{a} = \langle 1, 2 \rangle$.

Solution: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the vector

$$\mathbf{p}_a(b) = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(-\frac{2}{\sqrt{5}}\right) \frac{1}{\sqrt{5}} \langle 1, 2 \rangle.$$

We therefore obtain $\mathbf{p}_a(b) = -\left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$.



Example

Find the vector projection of $\mathbf{a} = \langle 1, 2 \rangle$ onto $\mathbf{b} = \langle -4, 1 \rangle$.

Solution: The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is the vector

$$\mathbf{p}_b(a) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}\right) \frac{\mathbf{b}}{|\mathbf{b}|} = \left(-\frac{2}{\sqrt{17}}\right) \frac{1}{\sqrt{17}} \langle -4, 1 \rangle.$$

We therefore obtain $\mathbf{p}_a(b) = \left\langle \frac{8}{17}, -\frac{2}{17} \right\rangle$.