mth 235 L 44

Plan: * Second order linear eqs. (Chp 3)

* First order linear and non-linear eqs. (Chp 2)

* Practice Final Exam
To be posted later this evening.

* Solutions will be posted on Monday.

* Review Chp 3: Second order linear eqs.

$$\boxed{y'' + a_1 y' + a_0 y = g(x)}$$

- Homogeneous Eqs. : $\boxed{g = 0}$

$y(x) = e^{rx}$, $\quad P(r) = r^2 + a_1 r + a_0 = 0$

(a) $r_1 \neq r_2$, real

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

(b) $r_1 \neq r_2$, complex , $r_\pm = \alpha \pm i\beta$

$$\boxed{y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)}$$

(c) $r_1 = r_2 = r$, real

$$\boxed{y(x) = (c_1 + c_2 x)\, e^{rx}}$$

* Recall: [Case (c) is solved using the Reduction of order method. See page 170 in text book. Do extra homework problems Sect. 3.4 : 23, 25, 27.]

- Non-homogeneous eqs: $\boxed{g \neq 0}$

(1) Undetermined coeffs. (guessing y_p)

(2) Variation of parameters.

$$\boxed{y_p = u_1 y_1 + u_2 y_2}$$

$$\boxed{u_1' = -\frac{y_2 g}{W}}, \quad \boxed{u_2' = \frac{y_1 g}{W}}.$$

* Example [knowing that $y_1(x) = x^2$ is Sol. of

$$x^2 y'' - 4x\, y' + 6y = 0, \quad x > 0,$$

find a second solution l.i. to y_1]

Sol:

- Notice: The eq. has variable coeffs.
- use: Reduction of order method.

[$y_1(x) = x^2$ is sol., since:

$$x^2(2) - 4x(2x) + 6x^2 = 0.$$]

We look for a Sol. y_2 of the form:

$$\boxed{y_2(x) = v(x)\, y_1(x)}$$

We find the eq. for v.

$$\left[\begin{aligned} y_2 &= x^2 v \\ y_2' &= x^2 v' + 2x\, v \\ y_2'' &= x^2 v'' + 4x\, v' + 2v \end{aligned}\right]$$

$$\left[\begin{aligned} &x^2 (x^2 v'' + 4 x v' + 2 v) \\ &- 4x (x^2 v' + 2x v) \\ &+ 6 (x^2 v) \qquad = 0 \end{aligned}\right]$$

$$x^4 v'' + 4 \underline{x^3} v' + 2 \underline{x^2} v$$
$$- 4 \underline{x^3} v' - 8 \underline{x^2} v \quad + 6 \underline{x^2} v = 0$$

$$x^4 v'' = 0 \quad \Rightarrow \quad \boxed{v'' = 0} \quad (x > 0)$$

$$\boxed{v(x) = c_1 + c_2 x}$$

$$\boxed{y_2 = c_1 y_1 + c_2 x y_1}$$

Choose $c_1 = 0$, $c_2 = 1$

$$\boxed{y_2(x) = x^3} \quad , \quad \boxed{y_1(x) = x^2}$$

* Example : Find the sol. to the IVP

$$y'' - 2y' - 3y = 3e^{-t}$$

$$y(0) = 1, \quad y'(0) = \frac{1}{4}.$$

Sol :

(1) Find sols. to the homogeneous eq.

$$y(x) = e^{rx}, \qquad p(r) = r^2 - 2r - 3 = 0$$

$$r_{\pm} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2.$$

$$\boxed{r_1 = 3}, \quad \boxed{r_2 = -1}$$

$$\boxed{y_1(t) = e^{3t}}, \quad \boxed{y_2(t) = e^{-t}}$$

(2) Guess the y_p.

$$\boxed{g(t) = 3e^{-t}} \Rightarrow \boxed{y_p = k e^{-t}.}$$

but : $\boxed{y_2 = e^{-t}}$ $\Rightarrow$

$$\boxed{y_p = k t e^{-t}}$$

(3) Find the undetermined coeff. k.

$$\boxed{y_p' = k e^{-t} - kt e^{-t}}$$

$$y_p'' = -k e^{-t} - k e^{-t} + k t e^{-t}$$

$$\boxed{y_p'' = -2k e^{-t} + kt e^{-t}}$$

$$\left[\begin{array}{l} (-2k e^{-t} + kt e^{-t}) \\ -2 (k e^{-t} - kt e^{-t}) \\ -3 (kt e^{-t}) = 3 e^{-t} \end{array} \right]$$

$$k e^{-t} [-2 + t - 2 + 2t - 3t] = 3 e^{-t}$$

$$-4k = 3 \Rightarrow \boxed{k = -\frac{3}{4}}$$

$$\boxed{y_p(t) = -\frac{3}{4} t e^{-t}}$$

(4) Find the general sol.

$$\boxed{y(t) = c_1 e^{3t} + c_2 e^{-t} - \frac{3}{4} t e^{-t}}$$

(5) Impose the I.C.

$$\boxed{y'(t) = 3 c_1 e^{3t} - c_2 e^{-t} - \frac{3}{4}\left(e^{-t} - t e^{-t} \right)}$$

$$\left.\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ \frac{1}{4} &= y'(0) = 3c_1 - c_2 - \frac{3}{4} \end{aligned}\right] \Rightarrow$$

$$\Rightarrow \boxed{\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 - c_2 &= 1 \end{aligned}}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-1-3} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}$$

$$\boxed{y(t) = \frac{1}{2}\left(e^{3t} + e^{-t}\right) - \frac{3}{4}\, t e^{-t}}$$

* <u>Review</u>: Variation of parameters problems, Sect. 3.6.

* Review Chp 2 : First order linear non-linear eqs.

- Linear first order eqs.

$$\boxed{y' + p(t)\, y = q(t)}$$

Integrating factor method: $\boxed{\mu(t) = e^{\int p(t)\,dt}}$

- Separable (non-linear) eqs.

$$\boxed{h(y)\, y'(t) = g(t)}$$

Integrate with the substitution:

$$\boxed{u = y(t)} \qquad \boxed{du = y'(t)\, dt}$$

The solution can be found in implicit or explicit form.

- Homogeneous eqs. can be converted into separable eqs.

  Read Page 49 in textbook.

- Modelling with first order eqs. (sect. 2.3)

- Bernoulli eqs. Read page 77 in the textbook.

$$\boxed{y' + p(t)\, y = q(t)\, y^n}$$

$\boxed{n: \text{positive integer.}}$

Bernoulli eqs for y can be converted into linear eqs. for

$$\boxed{v = \frac{1}{y^{n-1}}}$$

- Exact eqs. and integrating factors.

$$\boxed{N(x,y)\, y' + M(x,y) = 0.} \quad (1)$$

(1) is exact iff $\boxed{N_x = M_y}$

If (1) is exact, then there exist ψ sol. of:

$$\boxed{N = \psi_y}, \qquad \boxed{M = \psi_x}.$$

The sol. of (1) is given in implicit form as:

$$\boxed{\psi(x, y(x)) = c.}$$

* Example : [Find all solutions of

$$y' = \frac{x^2 + xy + y^2}{xy}$$]

Sol.

The sum of powers in x and y on every term is the same number, two in this case.

The equation is homogeneous. in y.

$$y' = \frac{x^2 + xy + y^2}{xy} \quad \frac{(1/x^2)}{(1/x^2)}$$

$$\boxed{y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}}$$

$$\boxed{v(x) = \frac{y(x)}{x}}$$

$$\boxed{y' = \frac{1 + v + v^2}{v}}$$

$$y = xv \quad \Rightarrow \quad \boxed{y' = xv' + v}$$

$$xv' + v = \frac{1 + v + v^2}{v}$$

$$xv' = \frac{1 + v + v^2}{v} - v$$

$$xv' = \frac{1 + v + v^2 - v^2}{v}$$

$$\boxed{xv' = \frac{1+v}{v}}$$

separable eq.

$$\boxed{\frac{v}{1+v} v' = \frac{1}{x}}$$

$$\int \frac{v \; v'}{1+v} dx = \int \frac{dx}{x} + c$$

Substitution: $u = 1+v$, $du = v'\,dx$

$$\int \frac{(u-1)}{u}\,du = \int \frac{dx}{x} + c$$

$$\int \left(1 - \frac{1}{u}\right) du = \int \frac{dx}{x} + c$$

$$\boxed{u - \ln|u| = \ln|x| + c}$$

$$\boxed{1+v - \ln|1+v| = \ln|x| + c}$$

$$\boxed{v = \frac{y}{x}}$$

$$\boxed{1 + \frac{y(x)}{x} - \ln\left|1 + \frac{y(x)}{x}\right| = \ln|x| + c}$$

* Example : $\left[\begin{array}{l} \text{Find the sol. to IVP} \\ y' + y + e^{2x}\, y^3 = 0 \\ y(0) = \frac{1}{3} \end{array} \right]$

Sol.

This is a Bernoulli eq.

$$\boxed{y' + y = -e^{2x}\, y^3} \quad , \; n = 3.$$

Divide by y^3 :

$$\boxed{\frac{y'}{y^3} + \frac{1}{y^2} = -e^{2x}}$$

$$\boxed{v = \frac{1}{y^2}} \quad \Rightarrow \quad v = y^{-2}$$

$$v' = -2\, y^{-3}\, y'$$

$$v' = -2\, \frac{y'}{y^3} \quad \Rightarrow$$

$$\boxed{\frac{y'}{y^3} = -\frac{1}{2}\, v'}$$

$$-\frac{1}{2} v' + v = -e^{2x}$$

$$\boxed{v' - 2v = 2e^{2x}}$$ linear eq. for v.

$$\boxed{\mu(x) = e^{-2x}}$$

$$e^{-2x} v' - 2e^{-2x} v = 2$$

$$\left(e^{-2x} v\right)' = 2$$

$$e^{-2x} v = 2x + c$$

$$\boxed{v(x) = (2x+c)\, e^{2x}}$$

$$\boxed{v = \frac{1}{y^2}}$$

$$\frac{1}{y^2} = (2x+c)\, e^{2x}$$

$$y^2 = \frac{1}{e^{2x}(2x+c)} \Rightarrow \boxed{y_{\pm}(x) = \pm\frac{1}{e^{x}\sqrt{2x+c}}}$$

I.C. $\quad y(0) = \frac{1}{3} \quad \Rightarrow$ $\boxed{\text{choose } y_{+}(x).}$

$$\frac{1}{3} = y_{+}(0) = \frac{1}{\sqrt{c}} \Rightarrow \boxed{c = 9}$$

$$\boxed{y(x) = \frac{e^{-x}}{\sqrt{2x+9}}}$$

* **Example** : [Find all sols. of

$$2xy^2 + 2y + 2x^2 y y' + 2x y' = 0$$]

Sol:

$$\underbrace{(2x^2 y + 2x)}_{N} y' + \underbrace{(2x y^2 + 2y)}_{M} = 0$$

$N = 2x^2 y + 2x \Rightarrow N_x = 4xy + 2$

$M = 2xy^2 + 2y \Rightarrow M_y = 4xy + 2$

Exact eq.

Find ψ sol. of : $\boxed{\psi_y = N}$ $\boxed{\psi_x = M}$

$\psi_y = N = 2x^2 y + 2x$

$$\boxed{\psi = x^2 y^2 + 2xy + g(x)}$$

$$\psi_x = 2xy^2 + 2y + g'(x) = M = 2xy^2 + 2y$$

$$\boxed{g'(x) = 0}$$

$$\boxed{g(x) = c}$$

$$\boxed{\psi(x,y) = x^2 y^2 + 2xy + c}$$

$$\boxed{x^2 [y(x)]^2 + 2x\, y(x) + c = 0}$$

(Implicit solution.)