mth 235 L 43

Plan:
- * Systems of Linear Eqs (Chp 7)
- * Laplace Transforms (Chp 6)
- * Power Series Sols. (Chp 5)
- * Second order linear eqs. (Chp 3)
- * First order diff. eqs. (Chp 2).

- Solutions to our exams 1, 2, 3 posted.
- Solutions to our make up exams 1, 3 also posted.
- Empty exams also posted.
- Use these solutions wisely.

* Review Chp 7: Systems of linear eqs.

Recall: Always find: $\begin{cases} \text{eigenvalues}, & \lambda_i, \\ \text{eigenvectors}, & \underline{v}^{(i)}, \end{cases}$

of the coefficient matrix. (real)

(a) $\lambda_1 \neq \lambda_2$ real $\Rightarrow \{\underline{v}^{(1)}, \underline{v}^{(2)}\}$ l.i.

$$\boxed{\underline{x}(t) = c_1 \underline{v}^{(1)} e^{\lambda_1 t} + c_2 \underline{v}^{(2)} e^{\lambda_2 t}}$$

(b) $\lambda_1 \neq \lambda_2$ complex.

Denoting: $\lambda_\pm \Rightarrow$ $\boxed{\begin{aligned} \lambda_\pm &= \alpha \pm i\beta \\ \underline{v}^{(\pm)} &= \underline{a} \pm i\underline{b} \end{aligned}}$

complex-valued sols.

$$\underline{x}^{(\pm)}(t) = (\underline{a} \pm i\underline{b})\, e^{(\alpha \pm i\beta)t}$$

$$\underline{x}^{\pm}(t) = e^{\alpha t}(\underline{a} \pm i\underline{b})(\cos(\beta t) \pm i \sin(\beta t))$$

$$\boxed{\begin{aligned} \underline{x}^{\pm}(t) &= e^{\alpha t}\left(\underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)\right) \\ &\quad \pm e^{\alpha t}\left(\underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)\right) i \end{aligned}}$$

Real-valued fund. sols.

$$\boxed{\begin{aligned} \underline{x}^{(1)}(t) &= e^{\alpha t}\left(\underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)\right) \\ \underline{x}^{(2)}(t) &= e^{\alpha t}\left(\underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)\right) \end{aligned}}$$

$$\boxed{\underline{x}(t) = c_1\,\underline{x}^{(1)}(t) + c_2\,\underline{x}^{(2)}(t)}$$ Real-valued general sol.

(c) $\lambda_1 = \lambda_2 = \lambda$ real

and only one $\underline{v}$.

Find $\underline{w}$ sol. of $(A - \lambda I)\underline{w} = \underline{v}$.

$$\boxed{\begin{aligned} \underline{x}^{(1)}(t) &= \underline{v}\,e^{\lambda t} \\ \underline{x}^{(2)}(t) &= (\underline{v}\,t + \underline{w})\,e^{\lambda t} \end{aligned}}$$

$$\boxed{\underline{x}(t) = c_1\,\underline{x}^{(1)}(t) + c_2\,\underline{x}^{(2)}(t)}$$

* **Example:** (Final Exam June 13, 2008)

Find the sol. to the IVP

$$\underline{x}'(t) = A\,\underline{x}(t), \quad A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Sol:

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1) - 8 = \lambda^2 - 1 - 8$$

$$P(\lambda) = \lambda^2 - 9 = 0 \quad \Rightarrow \quad \boxed{\lambda_\pm = \pm 3}$$

$\lambda_+ = 3.$

$$A - 3I = \begin{bmatrix} 1-3 & 4 \\ 2 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$v_1 = 2 v_2 \quad \Rightarrow \quad \boxed{\underline{v}^{(+)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda_+ = 3}$$

$\lambda_- = -3.$

$$A + 3I = \begin{bmatrix} 1+3 & 4 \\ 2 & -1+3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$v_1 = -v_2 \quad \Rightarrow \quad \boxed{\underline{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \lambda_- = -3.}$$

General solution:

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}.}$$

I.C.

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \underline{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(2+1)} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$$

$$\boxed{\underline{x}(t) = \frac{5}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}}$$

* Review Chp6 : Laplace Transform methods

Recall :

(18) $$\boxed{\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - \dots - f^{(n-1)}(0)}$$

(13) $$\boxed{e^{-cs} \mathcal{L}[f(t)](s) = \mathcal{L}[u(t-c) f(t-c)]}$$

(14) $$\boxed{\mathcal{L}[f(t)](s-c) = \mathcal{L}[e^{ct} f(t)]}$$

- Convolutions

$$\boxed{\mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)] \; \mathcal{L}[g(t)].}$$

- Partial fraction decompositions.

* **Example:** (Final Exam June 13, 2008)

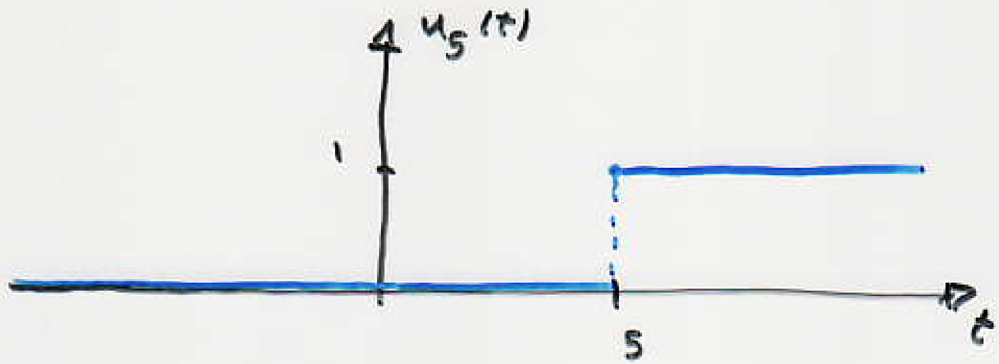
Use L.T. to find the sol. IVP

$$y'' + 9y = u_5(t)$$

$$y(0) = 3 \qquad y'(0) = 2$$

Sol:

Recall the **notation:** $u_5(t) = u(t-5)$



$$\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \mathcal{L}[u_5(t)]$$

$$\boxed{\mathcal{L}[y''] + 9\,\mathcal{L}[y] = \frac{e^{-5s}}{s}}$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)$$

$$\boxed{\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - 3s - 2}$$

$$(s^2+9)\,\mathcal{L}[y] - 3s - 2 = \frac{e^{-5s}}{s}$$

$$(s^2+9)\,\mathcal{L}[y] = 3s+2 + \frac{e^{-5s}}{s}$$

$$\boxed{\mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + e^{-5s}\,\frac{1}{s(s^2+9)}}$$

Partial fraction method.

$$\boxed{\frac{1}{s(s^2+9)} = \frac{a}{s} + \frac{bs+c}{(s^2+9)}}$$

$$1 = a(s^2+9) + (bs+c)s$$

$$1 = as^2 + 9a + bs^2 + cs$$

$$\boxed{1 = (a+b)s^2 + cs + 9a}$$

$a+b=0 \Rightarrow b=-a$

$c=0$

$9a=1 \Rightarrow \boxed{a=\frac{1}{9}}$

$\Rightarrow \boxed{b=-\frac{1}{9}}$ $\boxed{c=0}$

$$\boxed{\frac{1}{s(s^2+9)} = \frac{1}{9}\,\frac{1}{s} - \frac{1}{9}\,\frac{s}{(s^2+9)}}$$

$$\left[\ \mathcal{L}[y] = \frac{(3s+2)}{(s^2+9)} + \frac{1}{9}\,\frac{e^{-5s}}{s} - \frac{1}{9}\,e^{-5s}\frac{s}{(s^2+9)}\ \right]$$

$$\left[\ \begin{aligned} \mathcal{L}[y] &= 3\left(\frac{s}{s^2+9}\right) + \frac{2}{3}\left(\frac{3}{s^2+9}\right) \\ &\quad + \frac{1}{9}\left(\frac{e^{-5s}}{s}\right) - \frac{1}{9}\,e^{-5s}\left(\frac{s}{s^2+9}\right) \end{aligned}\ \right]$$

$$\boxed{\begin{aligned} \mathcal{L}[y] &= 3\ \mathcal{L}[\cos(3t)] + \frac{2}{3}\ \mathcal{L}[\sin(3t)] \\ &\quad + \frac{1}{9}\ \mathcal{L}[u(t-5)] - \frac{1}{9}\ e^{-5s}\ \mathcal{L}[\cos(3t)] \end{aligned}}$$

(13) $$\boxed{e^{-cs}\,\mathcal{L}[f(t)](s) = \mathcal{L}[u(t-c)\, f(t-c)]}$$

$$\boxed{e^{-5s}\,\mathcal{L}[\cos(3t)] = \mathcal{L}[u(t-5)\,\cos(3(t-5))]}$$

$$\mathcal{L}[Y] = 3\,\mathcal{L}[\cos(3t)] + \frac{2}{3}\,\mathcal{L}[\sin(3t)] + \frac{1}{9}\,\mathcal{L}[u(t-5)] - \frac{1}{9}\,\mathcal{L}[u(t-5)\cos(3(t-5))]$$

$$\boxed{Y(t) = 3\cos(3t) + \frac{2}{3}\sin(3t) + \frac{1}{9}u(t-5) - \frac{1}{9}u(t-5)\cos[3(t-5)].}$$

$$\boxed{Y(t) = 3\cos(3t) + \frac{2}{3}\sin(3t) + \frac{1}{9}u(t-5)\left[1-\cos(3(t-5))\right]}$$

* Review Chp 5 : Power Series Solutions

Recall :

$$\boxed{a(x)\, y'' + b(x)\, y' + c(x)\, y = 0}$$

(a) x_0 : regular point. $\Rightarrow$ $\boxed{y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n.}$

$\boxed{\text{recurrence rel. for } a_n.}$

(b) x_0 : regular-singular point. $\Rightarrow$ $\boxed{y(x) = \sum_{n=0}^{\infty} a_n \, (x-x_0)^{n+r}}$

$\boxed{\text{indicial eq. for } r, \text{ recurrence rel. for } a_n.}$

(c) Euler Eq. $\boxed{(x-x_0)^2 y'' + \alpha (x-x_0) y' + \beta y = 0}$

Solutions: $y(x) = |x-x_0|^r$

r sol. of indicial eq.

$$\boxed{q(r) = r(r-1) + \alpha r + \beta = 0}$$

(c1) $r_1 \neq r_2$, real

$$\boxed{y(x) = c_1 |x-x_0|^{r_1} + c_2 |x-x_0|^{r_2}}$$

(c2) $r_1 \neq r_2$ complex : $r_\pm = \lambda \pm i\mu$

$$\boxed{\begin{aligned} y(x) = {} & c_1 |x-x_0|^{\lambda} \cos(\mu \ln|x-x_0|) \\ & + c_2 |x-x_0|^{\lambda} \sin(\mu \ln|x-x_0|) \end{aligned}}$$

(c3) $r_1 = r_2 = r$ real

$$\boxed{y(x) = (c_1 + c_2 \ln|x-x_0|)\, |x-x_0|^{r}}$$

* *Example*: (Final Exam June 13, 2008)

Find the recurrence relation for the coefficients of a power series sol centered at $x_0 = 0$ of

$$y'' - 3y' + xy = 0. \qquad (1)$$

Sol:

$x_0 = 0$ regular point of (1).

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow \boxed{xy = \sum_{n=0}^{\infty} a_n x^{n+1}}$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{(n-1)} \Rightarrow \boxed{-3y' = \sum_{n=0}^{\infty} -3n a_n x^{n-1}}$$

$$\boxed{y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{(n-2)}}$$

$$\boxed{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (-3)n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{(n+1)} = 0}$$

$$\boxed{\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\substack{m = n-2 \\ m \to n}} + \underbrace{\sum_{n=1}^{\infty} (-3)n a_n x^{n-1}}_{\substack{m = n-1 \\ m \to n}} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}_{\substack{m = n+1 \\ m \to n}} = 0}$$

$$\boxed{\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} (-3)(n+1)\, a_{n+1}\, x^n + \sum_{n=1}^{\infty} a_{n-1}\, x^n = 0}$$

$$\boxed{(2)(1)\, a_2 + (-3)(1)\, a_1 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - 3(n+1)\, a_{n+1} + a_{n-1} \right] x^n = 0}$$

$$\boxed{2a_2 - 3a_1 = 0}$$

$$\boxed{(n+2)(n+1)\, a_{n+2} - 3(n+1)\, a_{n+1} + a_{n-1} = 0} \quad \boxed{n \geq 1}$$

* Review Chp 3 : Second Order Linear Eqs.

$$\boxed{y'' + a_1 y' + a_0 y = g(t).}$$

- Homogeneous eq. : $g = 0$.

$y(x) = e^{rx}$, $P(r) = r^2 + a_1 r + a_0 = 0$.

(a) $r_1 \neq r_2$, real

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

(b) $r_1 \neq r_2$, complex , $r_\pm = \alpha \pm i\beta$.

$$\boxed{y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)}$$

(c) $r_1 = r_2 = r$, real,

$$\boxed{y(x) = (c_1 + c_2 x) e^{rx}}$$

- Recall : [Case (c) is solved using the :

  Reduction of order method.

  See page 170 in textbook.

  Do extra homework problems:

  Sect. 3.4 : 23, 25, 27.]

- Non-homogeneous eqs.

  (1) Undetermined coefficients (guessing y_p)

  (2) Variation of parameters

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1' = -\frac{y_2 g}{W} \qquad u_2' = \frac{y_1 g}{W}.$$

* Review Chp 2: First Order Eqs.

- Linear first order eqs.

$$\boxed{y' + P(t)\, y = q(t)}$$

Integrating factor $\boxed{\mu = e^{\int P(t)\,dt}}$

- Separable eqs.

$$\boxed{h(y)\, y' = g(t)}$$

Integrate with substitution $\boxed{\begin{array}{l} u = y(t) \\ du = y'(t)\, dt \end{array}}$

- Homogeneous Eqs can be converted into separable eqs.

Read page 49 in textbook.

Example: $y' = \dfrac{x^2 + xy + y^2}{x^2 + y^2}$ is homogeneous.

$$y' = \frac{(x^2 + xy + y^2)}{(x^2 + y^2)} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$$

$$\boxed{y' = \frac{1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^2}}$$

$$\boxed{v = \frac{y}{x}} \quad \Rightarrow \quad y = x v$$

$$\boxed{y' = v + x v'}$$

$$v + x v' = \frac{1 + v + v^2}{1 + v^2}$$

$$\boxed{v' = \left[\frac{1 + v + v^2}{1 + v^2} - v\right] \frac{1}{x}}$$

separable eq. for v.

Solve for v.

compute y as follows : $\boxed{y = x\, v(x)}$

- Modelling with first order eqs. (Sect 2.3)

- Bernoulli eqs.

Read Page 77 in textbook.

$$\boxed{y' + p(t)\, y = q(t)\, y^n}$$

n positive integer

Bernoulli eqs. can be converted into a linear eq. for $\boxed{v = \frac{1}{y^{n-1}}}$

- Exact eqs. and integrating factors.

$$\boxed{N(x,y)\, y' + M(x,y) = 0}$$

Exact iff $\boxed{N_x = M_y}$

Then exists ψ sol. of

$$\boxed{\begin{aligned} N &= \psi_y \\ M &= \psi_x \end{aligned}}$$