mth 235 L 42

Plan: * Eve-periodic, odd-periodic extensions of functions.

* Fourier Series expansions

* Eigenvalue - Eigenfunctions Boundary value problems.

(Chptr 10.)

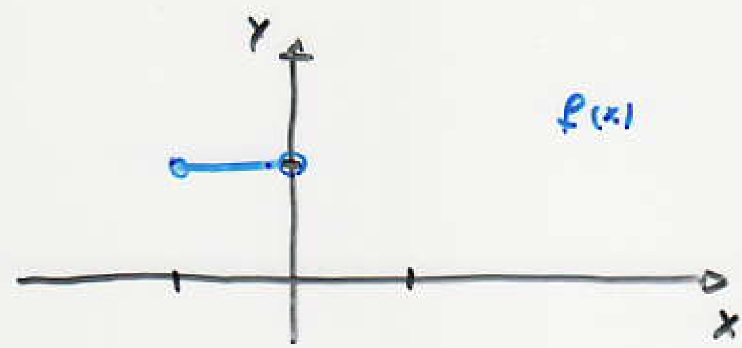
* <u>Sections:</u> [10.4 - 10.2] [Even-periodic, odd-periodic extensions of functions and Fourier series.]

* <u>Example:</u> [Graph the odd-periodic extension of

$$f(x) = 1 \quad , \quad x \in (-1, 0)$$

and then find the Fourier series of this extension.]

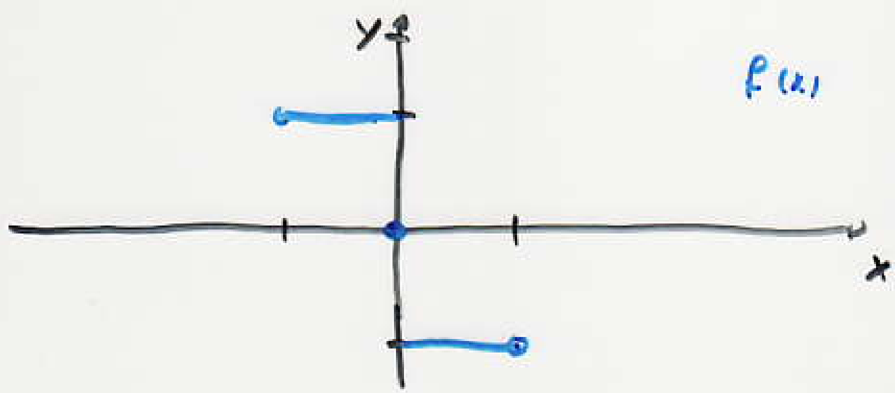
<u>Sol:</u>

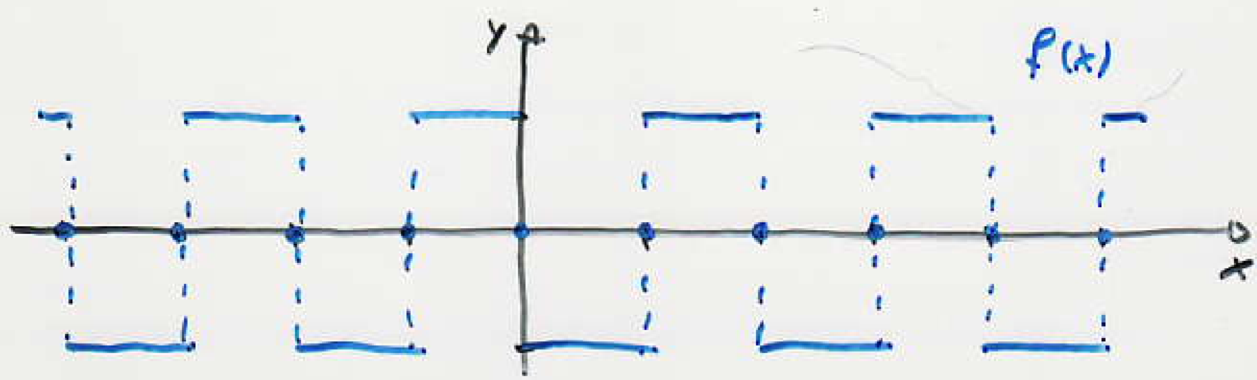


$f(-x) = -f(x)$

$f(0) = -f(0)$

$\boxed{f(0) = 0}$





$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x) \right]$$

$$f \text{ odd} \Rightarrow \boxed{a_n = 0} \quad \boxed{n = 0, 1, \ldots}$$

$$b_n = \frac{1}{1} \int_{-1}^{1} \underbrace{\underset{\text{odd}}{\underset{\uparrow}{f(x)}} \quad \underset{\text{odd}}{\underset{\uparrow}{\sin(n\pi x)}}}_{\text{even}} \, dx$$

$$b_n = 2 \int_0^1 f(x) \sin(n\pi x) \, dx$$

$$b_n = 2 \int_0^1 (-1) \sin(n\pi x) \, dx$$

$$b_n = -2 \left(\frac{-1}{n\pi} \right) \left[\cos(n\pi x) \right] \Big|_0^1$$

$$\boxed{b_n = \frac{2}{n\pi} \left[\cos(n\pi) - 1 \right]}$$

$$\boxed{b_n = \frac{2}{n\pi} \left[(-1)^n - 1 \right]}$$

If $n = 2k$, then $b_{2k} = \frac{2}{2k\pi}\left[(-1)^{2k} - 1\right]$

$$\boxed{b_{2k} = 0}$$

If $n = 2k+1$, then $b_{2k+1} = \frac{2}{(2k+1)\pi}\left[(-1)^{2k+1} - 1\right]$

$$\boxed{b_{2k+1} = -\frac{4}{(2k+1)\pi}}$$

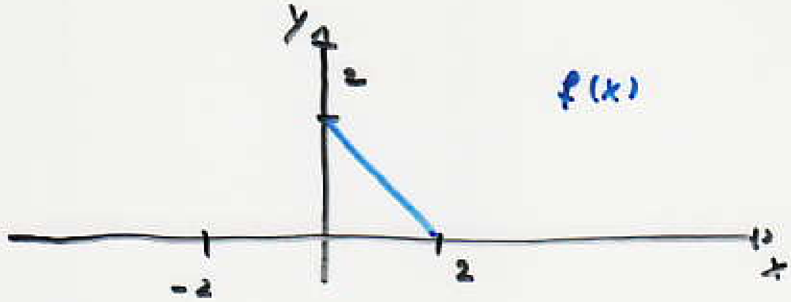
$$\boxed{f(x) = -\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin\left[(2k+1)\pi x\right].}$$

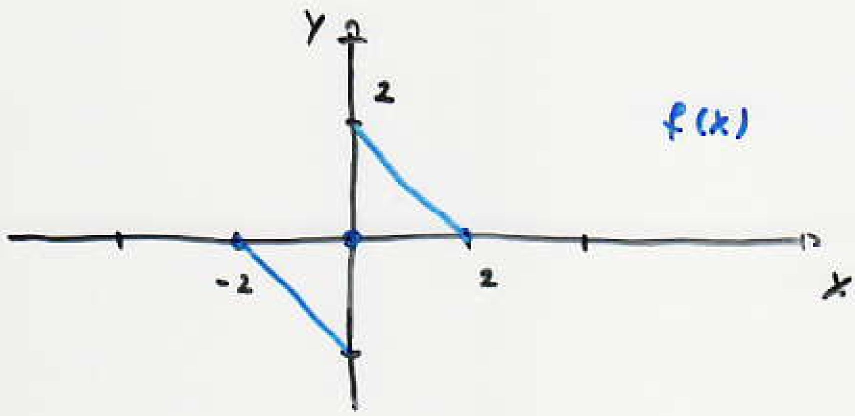
* Example : Graph the odd-periodic extension of

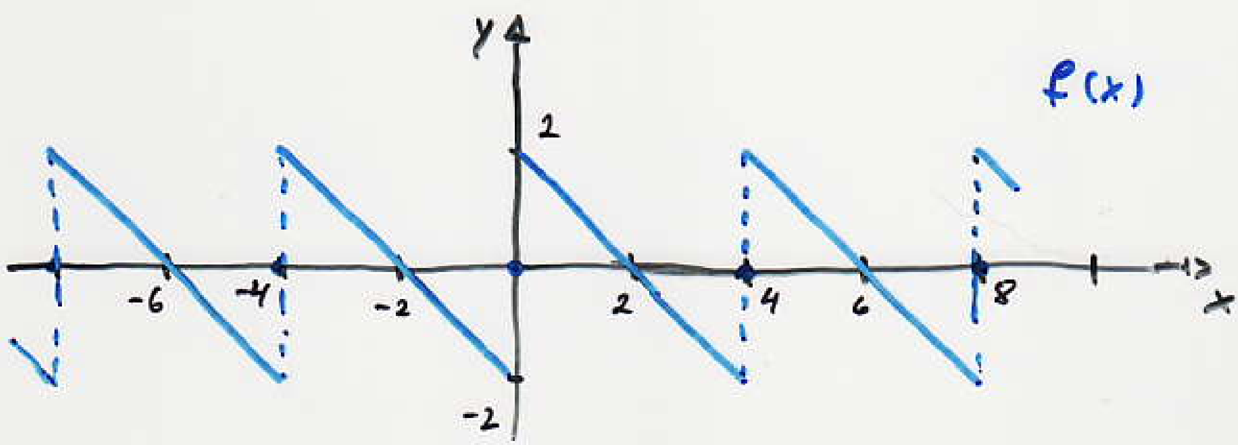
$$f(x) = 2 - x \ , \qquad x \in (0, 2)$$

and find the Fourier series of this extension.

Sol :







$T = 4$

$L = 2$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{2}x\right) + b_n \sin\left(\frac{n\pi}{2}x\right) \right]$$

f odd $\Rightarrow$ $\boxed{a_n = 0}$ $\boxed{n = 0, 1, \ldots}$

$$b_n = \frac{1}{2} \int_{-2}^{2} \underbrace{\underset{\text{odd}}{f(x)} \quad \underset{\text{odd}}{\sin\left(\frac{n\pi}{2}x\right)}}_{\text{even}} \, dx$$

$$b_n = \int_0^2 f(x) \sin\left(\frac{n\pi}{2}x\right) dx$$

$$b_n = \int_0^2 (2-x) \sin\left(\frac{n\pi}{2}x\right) dx$$

$$\boxed{b_n = 2 \int_0^2 \sin\left(\frac{n\pi}{2}x\right) - \int_0^2 x \sin\left(\frac{n\pi}{2}x\right) dx}$$

$$\boxed{\int \sin\left(\frac{n\pi}{2}x\right) dx = -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}x\right)}$$

$$\int x \sin\left(\frac{n\pi x}{2}\right) dx =$$

$$u = x \qquad v' = \sin\left(\frac{n\pi x}{2}\right)$$

$$u' = 1 \qquad v = -\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)$$

$$= -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \int \left(\frac{-2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$\boxed{\int x \sin\left(\frac{n\pi x}{2}\right) dx = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)}$$

$$b_n = 2\left[\left(-\frac{2}{n\pi}\right) \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2\right] +$$

$$+\left[\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]\Big|_0^2 - \left(\frac{2}{n\pi}\right)^2 \underbrace{\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2}_{=0}$$

$$b_n = -\frac{4}{n\pi}\left[\cos(n\pi) - 1\right]$$

$$+\left[\frac{4}{n\pi} \cos(n\pi) - 0\right] - 0$$

$$b_n = -\frac{4}{n\pi}\cos(n\pi) + \frac{4}{n\pi} + \frac{4}{n\pi}\cos(n\pi)$$

$$\boxed{b_n = \frac{4}{n\pi}}$$

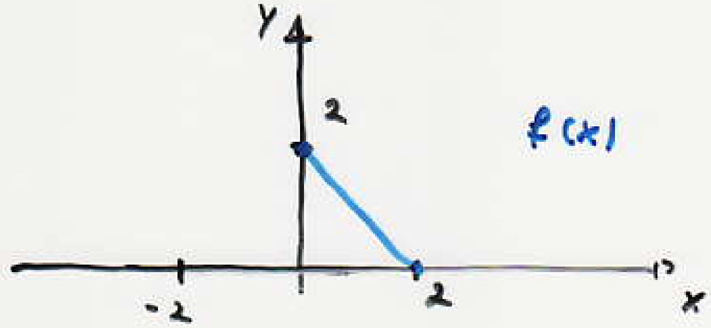
$$\boxed{f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}x\right)}$$

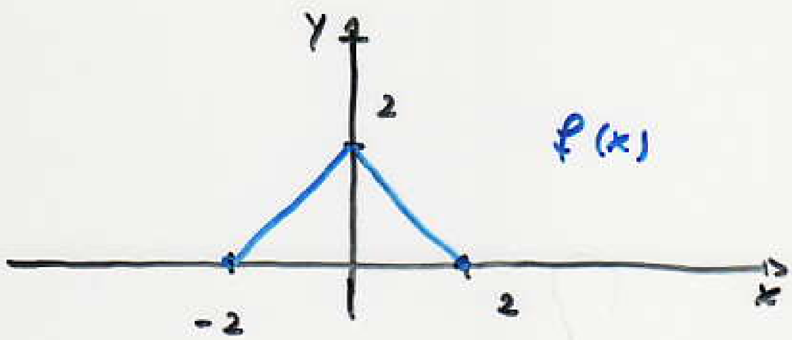
* <u>Example</u> : Graph the even-periodic extension of

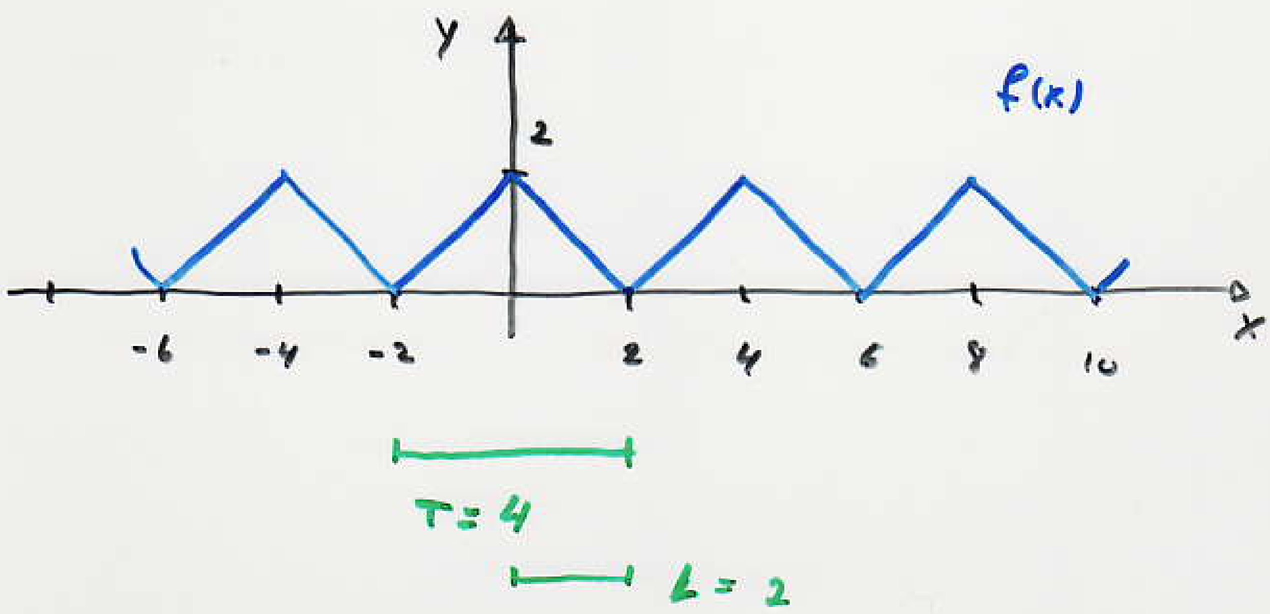
$$f(x) = 2 - x \quad , \quad x \in [0,2],$$

and find the Fourier series of this extension.

<u>Sol</u>:







$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{2} x\right) + b_n \sin\left(\frac{n\pi}{2} x\right) \right]$$

f is even $\Rightarrow$ $\boxed{b_n = 0}$ $\boxed{n = 1, 2, \ldots}$

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x)\,dx$$

$$a_0 = \frac{1}{2}\left(\frac{b\,h}{2}\right) 2 = \frac{1}{2} \frac{(2 \times 2)}{2}\, 2 = \frac{4}{2} \Rightarrow \boxed{a_0 = 2}$$

(area of triangles.)

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{n\pi}{2} x\right) dx$$

$$\underbrace{\underset{\text{even}}{f} \qquad \underset{\text{even}}{f}}_{\text{even}}$$

$$a_n = \int_{0}^{2} f(x) \cos\left(\frac{n\pi}{2} x\right) dx$$

$$a_n = \int_{0}^{2} (2 - x) \cos\left(\frac{n\pi}{2} x\right) dx$$

$$\boxed{a_n = 2 \int_{0}^{2} \cos\left(\frac{n\pi}{2} x\right) dx - \int_{0}^{2} x \cos\left(\frac{n\pi}{2} x\right) dx}$$

$$\boxed{\int \cos\left(\frac{n\pi}{2}x\right) dx = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}x\right)}$$

$$\int x \cos\left(\frac{n\pi}{2}x\right) dx =$$

$u = x \qquad v' = \cos\left(\frac{n\pi}{2}x\right)$

$u' = 1 \qquad v = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}x\right)$

$$= \frac{2x}{n\pi} \sin\left(\frac{n\pi}{2}x\right) - \frac{2}{n\pi} \int \sin\left(\frac{n\pi}{2}x\right) dx$$

$$\boxed{\int x \cos\left(\frac{n\pi}{2}x\right) dx = \frac{2x}{n\pi} \sin\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi}{2}x\right)}$$

$$a_n = 2\left[\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}x\right)\Big|_0^2\right]$$

$$- \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi}{2}x\right)\right]_0^2 - \left(\frac{2}{n\pi}\right)^2 \left[\cos\left(\frac{n\pi}{2}x\right)\Big|_0^2\right]$$

$$= 0 - 0 - \left(\frac{2}{n\pi}\right)^2 \left[\cos(n\pi) - 1\right]$$

$$\boxed{a_n = \left(\frac{2}{n\pi}\right)^2 \left[1 - \cos(n\pi)\right]}$$

$$\boxed{a_n = \left(\frac{2}{n\pi}\right)^2 \left[1 - (-1)^n\right]}$$

If $n = 2k$, then $\boxed{a_{2k} = 0}$

If $n = 2k+1$, then $\boxed{a_{2k+1} = \frac{4}{(2k+1)^2 \pi^2} \quad (2)}$

$$\boxed{f(x) = 1 + \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left((2k+1)\frac{\pi}{2} x\right)}$$

* Section (10.1) : Eigenvalues - Eigenfunctions Boundary Value problems.

* Example :

$$\left[\begin{array}{l}\text{Find the positive eigenvalues and their eigenfunctions of:}\\ y'' + \lambda y = 0 \\ y(0) = 0, \quad y(8) = 0\end{array}\right]$$

sol :

$\lambda > 0 \Rightarrow \lambda = \mu^2, \quad \mu > 0,$

$y(x) = e^{rx}, \quad P(r) = r^2 + \mu^2 = 0$

$$\boxed{r_{\pm} = \pm \mu i}$$

General sol.:

$$\boxed{y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)}$$

B.C.

$0 = y(0) = c_1 \Rightarrow \boxed{c_1 = 0}$

$$\boxed{y(x) = c_2 \sin(\mu x)}$$

$$0 = Y(8) = C_2 \sin(\mu 8)$$

$$C_2 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0$$

$$\mu_n 8 = n\pi$$

$$\boxed{\mu_n = \frac{n\pi}{8}}$$

$$\boxed{\lambda_n = \left(\frac{n\pi}{8}\right)^2}$$

$$\boxed{Y_n(x) = \sin\left(\frac{n\pi}{8}x\right)}$$

$$\boxed{n = 1, 2, \ldots}$$

* <u>Example</u> : Find the positive eigenvalues and their eigenfunctions of

$$y'' + \lambda y = 0$$

$$y(0) = 0 \quad , \quad y'(8) = 0$$

<u>Sol:</u>

$\lambda > 0 \quad , \qquad \lambda = \mu^2 \qquad \mu > 0. \qquad y(x) = e^{rx}$

$P(r) = r^2 + \mu^2 = 0 \quad \Rightarrow \quad r_{\pm} = \pm \mu i \quad ,$

$$\boxed{y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)}$$

BC: $\quad 0 = y(0) = c_1 \quad \Rightarrow \quad \boxed{c_1 = 0}$

$$\boxed{y(x) = c_2 \sin(\mu x)}$$

$$\boxed{y'(x) = \mu c_2 \cos(\mu x)}$$

$0 = y'(8) = \mu c_2 \cos(\mu 8) \quad ;$

$\mu > 0, \quad c_2 \neq 0, \quad \Rightarrow \quad \cos(\mu 8) = 0$

$$\mu_n 8 = (2n+1)\frac{\pi}{2}$$

$$\boxed{\mu_n = (2n+1)\frac{\pi}{16}}$$

$$\boxed{\lambda_n = (2n+1)^2\left(\frac{\pi}{16}\right)^2}$$

$$\boxed{y_n(x) = \sin\left((2n+1)\frac{\pi}{16}x\right)}$$

$$\boxed{n = 0, 1, 2, \ldots}$$

* <u>Example</u> : Find the non-negative eigenvalues and their eigenfunctions of

$$y'' + \lambda y = 0$$

$$y'(0) = 0 \,, \quad y'(8) = 0 \,.$$

<u>Sol:</u>

$\lambda \geq 0$, $\lambda = \mu^2$, $\mu \geq 0$, $y(x) = e^{rx}$

$$P(r) = r^2 + \mu^2 = 0 \Rightarrow \boxed{r_{\pm} = \pm \mu i}$$

case $\mu > 0$.

$$\boxed{y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).}$$

$$\boxed{y'(x) = -c_1 \mu \sin(\mu x) + \mu c_2 \cos(\mu x).}$$

$$0 = y'(0) = \mu c_2 \,, \quad \mu > 0 \Rightarrow \boxed{c_2 = 0}$$

$$\boxed{y(x) = c_1 \cos(\mu x)}$$

$$\boxed{y'(x) = -\mu c_1 \sin(\mu x)}$$

$$0 = Y'(8) = -\mu c_1 \sin(\mu 8)$$

$\mu > 0, \quad c_1 \neq 0 \quad \Rightarrow \quad \sin(\mu 8) = 0$

$$\mu_n 8 = n\pi$$

$$\boxed{\mu_n = \frac{n\pi}{8}}$$

$$\boxed{\lambda_n = \left(\frac{n\pi}{8}\right)^2} \qquad \boxed{Y_n(x) = \cos\left(\frac{n\pi}{8} x\right)}$$

$\boxed{n = 1, 2, \ldots}$

case $\mu = 0$

$$\boxed{Y(x) = c_1 + c_2 x.}$$

$$\boxed{Y'(x) = c_2.}$$

$$\left.\begin{array}{l} 0 = Y'(0) = c_2 \\ 0 = Y'(8) = c_2 \end{array}\right] \Rightarrow \boxed{c_2 = 0}$$

$$\boxed{\lambda_0 = 0} \qquad \boxed{Y_0 = 1}$$

* *Example* :

$$\left[\begin{array}{l} \text{Find the sol. of BVP} \\ y'' + y = 0 \\ y'(0) = 1 , \quad y(\tfrac{\pi}{3}) = 0 . \end{array}\right]$$

Sol.

$$y(x) = e^{rx} , \quad p(r) = r^2 + 1 = 0 \Rightarrow \boxed{r_{\pm} = \pm i}$$

$$\boxed{y(x) = c_1 \cos(x) + c_2 \sin(x)}$$

B.C.

$$y'(x) = - c_1 \sin(x) + c_2 \cos(x)$$

$$1 = y'(0) = c_2 \Rightarrow \boxed{c_2 = 1}$$

$$\boxed{y(x) = c_1 \cos(x) + \sin(x)}$$

B.C.

$$0 = y(\pi/3) = c_1 \cos(\pi/3) + \sin(\pi/3)$$

$$c_1 = - \frac{\sin(\pi/3)}{\cos(\pi/3)} = - \frac{\sqrt{3}/2}{1/2} \Rightarrow \boxed{c_1 = -\sqrt{3}}$$

$$\boxed{y(x) = - \sqrt{3} \cos(x) + \sin(x)}$$