mth 235 L 41

Plan: * Review : The separation of variables method for PDE.

* Examples: - The heat eq.

- The wave eq.

(10.5)

Final Exam :

- covers all sections
- ~ 15 problems
- ~ 3 Multiple choice

* Review: The Separation of Variables method for PDE.

Problem: solve an IBVP for a PDE.

unknown: $u(t,x)$.

Propose:

$$u(t,x) = \sum_{n=1}^{\infty} c_n \quad V_n(t) \quad W_n(x)$$

[Separation of variables method] [Fourier Series coeff.] [IVP for an ODE] [BVP e-value e-function Problem]

or

$$u(t,x) = \sum_{n=1}^{\infty} \tilde{V}_n(t) \quad W_n(x)$$

[General Sol. of ODE] [BVP e-value e-function problem.]

[IC and Fourier series fix the arbitrary consts. c_n.]

* Example : Find the sol. to the IBVP. for the heat eq.

$$4 \frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) \qquad x \in [0,2] \quad t > 0$$

$$u(0,x) = 3 \sin\left(\frac{\pi}{2} x\right)$$

$$u(t,0) = 0, \quad u(t,2) = 0.$$

Sol.

This is a P.D.E. : Heat eq.

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} c_n \, V_n(t) \, W_n(x)}$$

separation of variables

$$u_n(t,x) = V_n(t) \, W_n(x)$$

$$\boxed{4 \frac{\partial}{\partial t} u_n(t,x) = \frac{\partial^2}{\partial x^2} u_n(t,x)}$$

$$4 \, W_n(x) \, V_n'(t) = V_n(t) \, W_n''(x).$$

Divide the eq. by $V_n(t)\, W_n(x)$.

$$\frac{1}{V_n(t)\, W_n(x)}\, 4\, W_n(x)\, V_n'(t) = \frac{1}{V_n(t)\, W_n(x)}\, V_n(t)\, W_n''(x)$$

$$\boxed{4\,\underbrace{\frac{V_n'(t)}{V_n(t)}}_{\text{only } t} = \underbrace{\frac{W_n''(x)}{W_n(x)}}_{\text{only } x.}}$$

$$\boxed{4\,\frac{V_n'(t)}{V_n(t)} = -\lambda_n}$$

$$\boxed{\frac{W_n''(x)}{W_n(x)} = -\lambda_n}$$

$$\boxed{V_n'(t) + \frac{\lambda_n}{4} V_n(t) = 0} \qquad \boxed{W_n''(x) + \lambda_n W_n(x) = 0}$$

BVP:
$$\begin{bmatrix} w_n''(x) + \lambda_n\, w_n(x) = 0 \\ w_n(0) = 0 \quad , \quad w_n(2) = 0 \end{bmatrix}$$

IVP:
$$\begin{bmatrix} V_n'(t) + \frac{\lambda_n}{4}\, V_n(t) = 0 \\ V_n(0) = 1 \end{bmatrix}$$

Sol. of BVP

From sect. (10.1) we know: $\boxed{\lambda_n > 0.}$

Denote $\boxed{\lambda_n = (\mu_n)^2}$, $\mu_n > 0$.

$$w_n''(x) + \mu_n^2\, w_n(x) = 0$$

Sols. : $w_n(x) = e^{rx}$.

$$P(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad \boxed{r_\pm = \pm i \mu_n.}$$

general solution:

$$\boxed{w_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)}$$

BC: $0 = W_n(0) = c_1 \Rightarrow \boxed{c_1 = 0}$

so: $\boxed{W_n(x) = c_2 \sin(\mu_n x)}$

$$0 = W_n(2) = c_2 \sin(\mu_n 2)$$

$c_2 \neq 0. \Rightarrow \sin(\mu_n 2) = 0$

$$\boxed{\mu_n 2 = n\pi}$$

$$\mu_n = \frac{n\pi}{2} \Rightarrow$$

$\Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{2}\right)^2}$ $\boxed{W_n(x) = \sin\left(\frac{n\pi}{2}x\right)}$

Sol. of IVP. $\left[\begin{array}{c} V_n'(t) + \frac{\lambda_n}{4} V_n(t) = 0 \\ V_n(0) = 1 \end{array}\right]$

Integrating factor : $\boxed{\mu(t) = e^{\lambda_n t/4}}$

$$e^{\lambda_n t/4} V_n'(t) + \frac{\lambda_n}{4} e^{\lambda_n t/4} V_n(t) = 0$$

$$\left(e^{\lambda_n t/4} V_n(t) \right)' = 0$$

$$e^{\lambda_n t/4} V_n(t) = \tilde{c}_n \Rightarrow \boxed{V_n(t) = \tilde{c}_n e^{-\lambda_n t/4}}$$

I.C. : $1 = V_n(0) = \tilde{c}_n \Rightarrow \boxed{\tilde{c}_n = 1}$

$$\boxed{V_n(t) = e^{-\lambda_n t/4}} \qquad \lambda_n = \frac{n^2\pi^2}{4}$$

$$\boxed{V_n(t) = e^{-\frac{n^2\pi^2}{16} t}}$$

Sol $$\boxed{u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-\frac{n^2\pi^2}{16}t} \sin\left(\frac{n\pi}{2}x\right)}$$

BC for u: $$0 = u(t,0) = \sum_{n=1}^{\infty} (\;) \sin(0) = 0 \checkmark$$

$$0 = u(t,2) = \sum_{n=1}^{\infty} (\;) \sin(n\pi) = 0 \checkmark$$

IC for u:

$$\boxed{3\sin\left(\frac{\pi}{2}x\right) = u(0,x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2}x\right).}$$

The orthogonality of sine functions implies:

$$\left[\begin{aligned} &3\int_0^2 \sin\left(\frac{\pi}{2}x\right)\sin\left(m\frac{\pi}{2}x\right) dx = \\ &= \sum_{n=1}^{\infty} c_n \int_0^2 \sin\left(\frac{n\pi}{2}x\right)\sin\left(\frac{m\pi}{2}x\right) dx \end{aligned}\right]$$

If $m \neq 1$, then:

$$0 = c_m \left(\frac{2}{2}\right) \Rightarrow \boxed{c_m = 0, \quad m \neq 1}$$

If $m = 1$, then:

$$3 \left(\frac{2}{2}\right) = c_1 \left(\frac{2}{2}\right) \Rightarrow \boxed{c_1 = 3}$$

$$\boxed{u(t,x) = 3 e^{-\frac{\pi^2}{16} t} \sin\left(\frac{\pi}{2} x\right)}$$

* **Remark**: Not every PDE can be solved with the separation of variable method.

* **Example**: Determine whether the separation of variables method can be used in

$$\left(t+\frac{x}{c}\right)\frac{\partial^2}{\partial t^2}u(t,x) + k\,\frac{\partial^2 u(t,x)}{\partial x^2} = 0.$$

Sol:

If $u(t,x) = v(t)\,w(x)$, then:

$$\left(t+\frac{x}{c}\right) w(x)\, v''(t) = -k\, v(t)\, w''(x)$$

$$\boxed{\underbrace{-\frac{1}{k}\left(t+\frac{x}{c}\right)\frac{v''(t)}{v(t)}}_{t \text{ and } x} = \underbrace{\frac{w''(x)}{w(x)}}_{\text{only } x}}$$

NO.

* Remark: The separation of variables method works on PDE other than the heat eq.

* Example: Find the solution to the IBVP for the wave eq.

$$\boxed{\frac{\partial^2}{\partial t^2} u(t,x) = V_0^2 \frac{\partial^2}{\partial x^2} u(t,x)}$$

$x \in [0, 3\pi]$, $t \in [0, \infty)$, $V_0 > 0$.

IC: $\begin{bmatrix} u(0,x) = \sin(x) \\ \frac{\partial}{\partial t} u(0,x) = 0 \end{bmatrix}$

BC: $\begin{bmatrix} u(t,0) = 0 \\ u(t,3\pi) = 0. \end{bmatrix}$

Remarks:

- This is not the heat eq. (Fourier)
- This is a wave eq. (Bernoulli)
- It describes waves on a String.

  (sound on air)

  (etc.)
- There are two initial conditions:
  - Initial position of the string.
  - Initial velocity of the string.

Sol:

Let: $$u(t,x) = \sum_{n=1}^{\infty} \tilde{V}_n(t) \quad W_n(x)$$

[separation of variables.] [General sol. of an ODE] [BVP e-val. e-funct.]

Denote $u_n(t,x) = \tilde{V}_n(t) \quad W_n(x)$.

$$\boxed{W_n(x) \; \tilde{V}_n''(t) = v_0^2 \; \tilde{V}_n(t) \; W_n''(x)}$$

$$\boxed{\frac{1}{V_0^2} \frac{\tilde{V}_n''(t)}{\tilde{V}_n(t)} = \frac{W_n''(x)}{W_n(x)}}$$

only t only x ✓

$$\boxed{\frac{1}{V_0^2} \frac{\tilde{V}_n''(t)}{\tilde{V}_n(t)} = -\lambda_n}$$

$$\boxed{\frac{W_n''(x)}{W_n(x)} = -\lambda_n}$$

$$\boxed{\tilde{V}_n''(t) + \lambda_n V_0^2 V_n(t) = 0}$$

$$\boxed{W_n''(x) + \lambda_n W_n(x) = 0}$$

BVP : $\left[\begin{array}{l} W_n''(x) + \lambda_n W_n(x) = 0 \\ W_n(0) = 0, \quad W_n(3\pi) = 0 \end{array}\right]$

General sol. of : $\left[\tilde{V}_n''(t) + V_0^2 \lambda_n \tilde{V}_n(t) = 0 \right]$

Sol. of BVP:

From (10.1) we know $\lambda_n > 0$.

Denote $\boxed{\lambda_n = \mu_n^2,}$ $\mu_n > 0$.

Sols : $W_n(x) = e^{rx}$

$P(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad \boxed{r_\pm = \pm \mu_n i}$

General Sol:

$$\boxed{W_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)}$$

BC : $0 = W_n(0) = c_1 \Rightarrow \boxed{c_1 = 0}$

$$\boxed{W_n(x) = c_2 \sin(\mu_n x)}$$

$$0 = W_n(3\pi) = c_2 \sin(\mu_n 3\pi)$$

$c_2 \neq 0 \quad \Rightarrow \quad \sin(\mu_n 3\pi) = 0$

$\mu_n 3\pi = n\pi \quad \Rightarrow \quad \boxed{\mu_n = \frac{n}{3}}$

$\boxed{\lambda_n = \frac{n^2}{9}} \qquad \boxed{W_n(x) = \sin\left(\frac{nx}{3}\right)}$

General sol. of. $\boxed{\tilde{V}_n''(t) + v_0^2 \mu_n^2 \tilde{V}_n(t) = 0}$

second order ODE.

$\tilde{V}_n(t) = e^{rt} \quad \Rightarrow \quad P(r) = r^2 + v_0^2 \mu_n^2 = 0$

$\Rightarrow \boxed{r_\pm = \pm v_0 \mu_n i}$

$$\boxed{\tilde{V}_n(t) = c_n \cos(v_0 \mu_n t) + d_n \sin(v_0 \mu_n t)}$$

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} \left[c_n \cos\left(v_0 \frac{n}{3} t\right) + d_n \sin\left(v_0 \frac{n}{3} t\right)\right] \sin\left(\frac{n}{3}x\right)}$$

BC:

$$0 = u(t,0) = \sum_{n=1}^{\infty} [\quad] \sin(0) = 0 \checkmark$$

$$0 = u(t,3\pi) = \sum_{n=1}^{\infty} [\quad] \sin(n\pi) = 0 \checkmark$$

IC:

$$\sin(x) = u(0,x) \qquad (1)$$

$$0 = u'(0,x) \qquad (2)$$

$$\boxed{u'(t,x) = \sum_{n=1}^{\infty} \left[- c_n \sin\left(v \frac{n}{3} t\right) + d_n \cos\left(v_0 \frac{n}{3} t\right)\right] v_0 \frac{n}{3} \sin\left(\frac{n}{3}\right)}$$

$$\boxed{\sin(x) = u(0,x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{nx}{3}\right)} \qquad (1)$$

$$\boxed{0 = u'(0,x) = \sum_{n=1}^{\infty} d_n \, v_0 \frac{n}{3} \sin\left(\frac{nx}{3}\right).} \qquad (2)$$

The orthogonality of sine functions imply:

Eq. (1)

$$\boxed{\int_0^{3\pi} \sin(x)\,\sin\left(\frac{mx}{3}\right) dx = \sum_{n=1}^{\infty} c_n \int_0^{3\pi} \sin\left(\frac{nx}{3}\right) \sin\left(\frac{mx}{3}\right) dx}$$

- If $m \neq 3$, then

$$0 = c_m \left(\frac{3\pi}{2}\right) \Rightarrow \boxed{c_m = 0, \quad m \neq 3}$$

- If $m = 3$, then

$$\frac{3\pi}{2} = c_3 \frac{3\pi}{2} \Rightarrow \boxed{c_3 = 1}$$

Eq. (2)

$$\boxed{0 = \sum_{n=1}^{\infty} d_n \int_0^{3\pi} \sin\left(\frac{nx}{3}\right) \sin\left(\frac{mx}{3}\right) dx}$$

$$0 = d_m \frac{3\pi}{2} \Rightarrow \boxed{d_m = 0} \quad m \geq 1.$$

Recalling:

$$u(t,x) = \sum_{n=1}^{\infty} \left[c_n \cos(v_0 \mu_n t) + d_n \sin(v_0 \mu_n t) \right] \sin\left(\frac{n x}{3}\right)$$

then:

$$\boxed{u(t,x) = \cos(v_0 \mu_3 t) \sin\left(\frac{3x}{3}\right)}$$

$$\mu_n = \frac{n}{3} \Rightarrow \mu_3 = 1$$

$$\boxed{u(t,x) = \cos(v_0 t) \sin(x).}$$