mth 235 L 40

Plan:
* The Heat Equation
* Initial-Boundary Value Problem.
* Separation of variables method.
* Example.

(10.5)

Common Final Exam

Tuesday May 4, 10:00 am - 12:00 pm

Sects 09, 12 : In N130 BCC
(Business College Complex)

Sects 10, 11 : 1345 EB (our Lecture room)

Common Make-up Final Exam

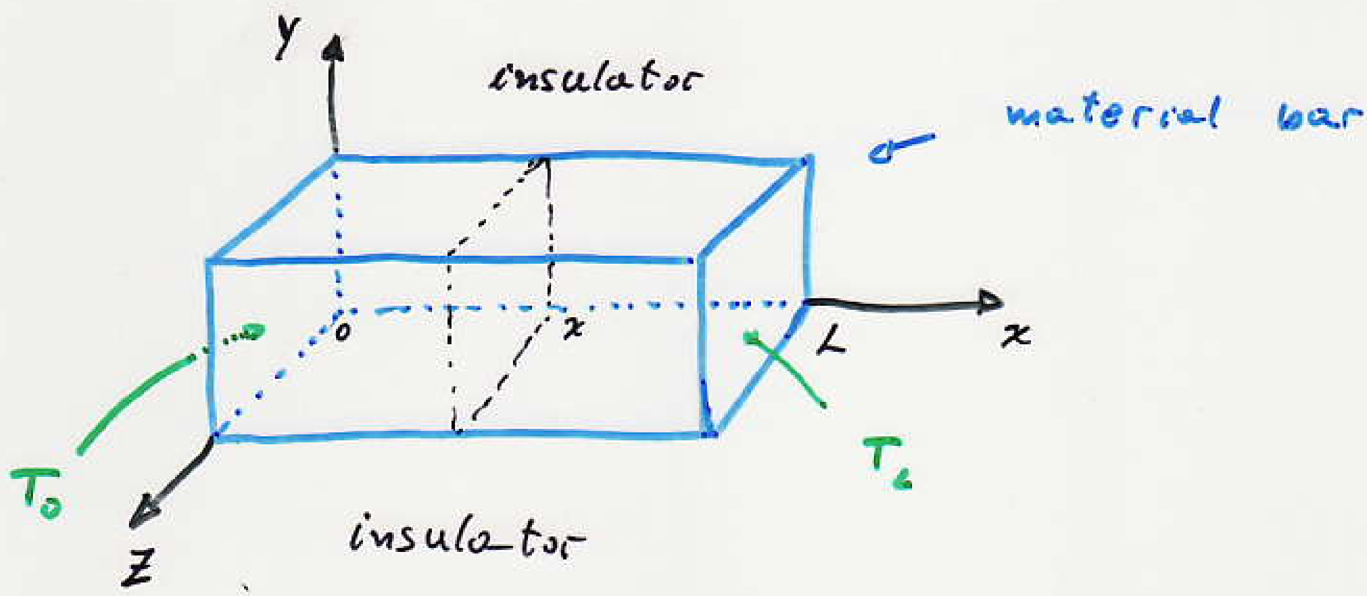
Wednesday May 5, 10:00 am - 12:00 pm

In 147 Communications Art Building.

* The Heat Equation.

- It describes the heat transport in a solid material.

- The physical situation.



- Simplification: The heat transfer occurs only along the x-axis

- The unknown of the problem is:

$u(t,x)$: Temperature of the bar at time t and position x.

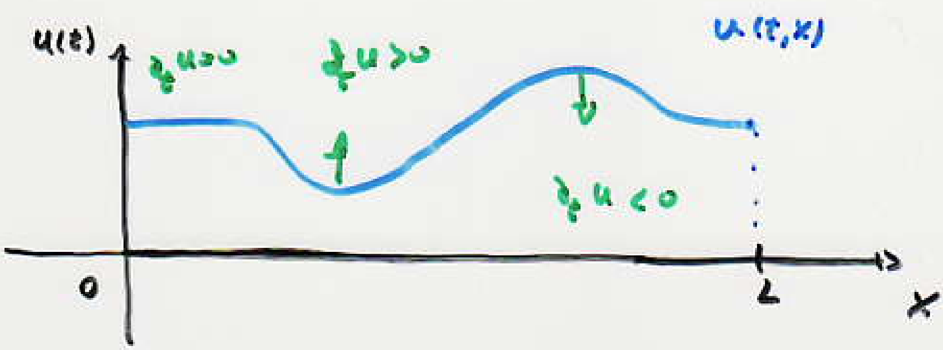
- The temperature u does NOT depend on y nor z.

- The (1-dimensional) heat equation is:

$$\boxed{\frac{\partial}{\partial t} u(t,x) = k \frac{\partial^2}{\partial x^2} u(t,x)} \tag{3.1}$$

$k > 0$, constant, (heat conductivity) $\qquad [k] = \dfrac{(\text{distance})^2}{(\text{time})}$

- Eq. (3.1) is a partial differential eq. (PDE)



t : constant

* <u>The</u> <u>Initial-Boundary</u> <u>Value</u> <u>Problem</u> (IBVP) <u>for</u> <u>the</u> <u>1-dimensional</u> <u>heat</u> <u>equation</u>.

Given a const. $k > 0$ and a funct. $f : [0, L] \to \mathbb{R}$ with $f(0) = f(L) = 0$, find $u : [0,\infty) \times [0, L] \to \mathbb{R}$ sol. of

$$\boxed{u : [0, \infty) \times [0, L] \to \mathbb{R}}$$

solution of:

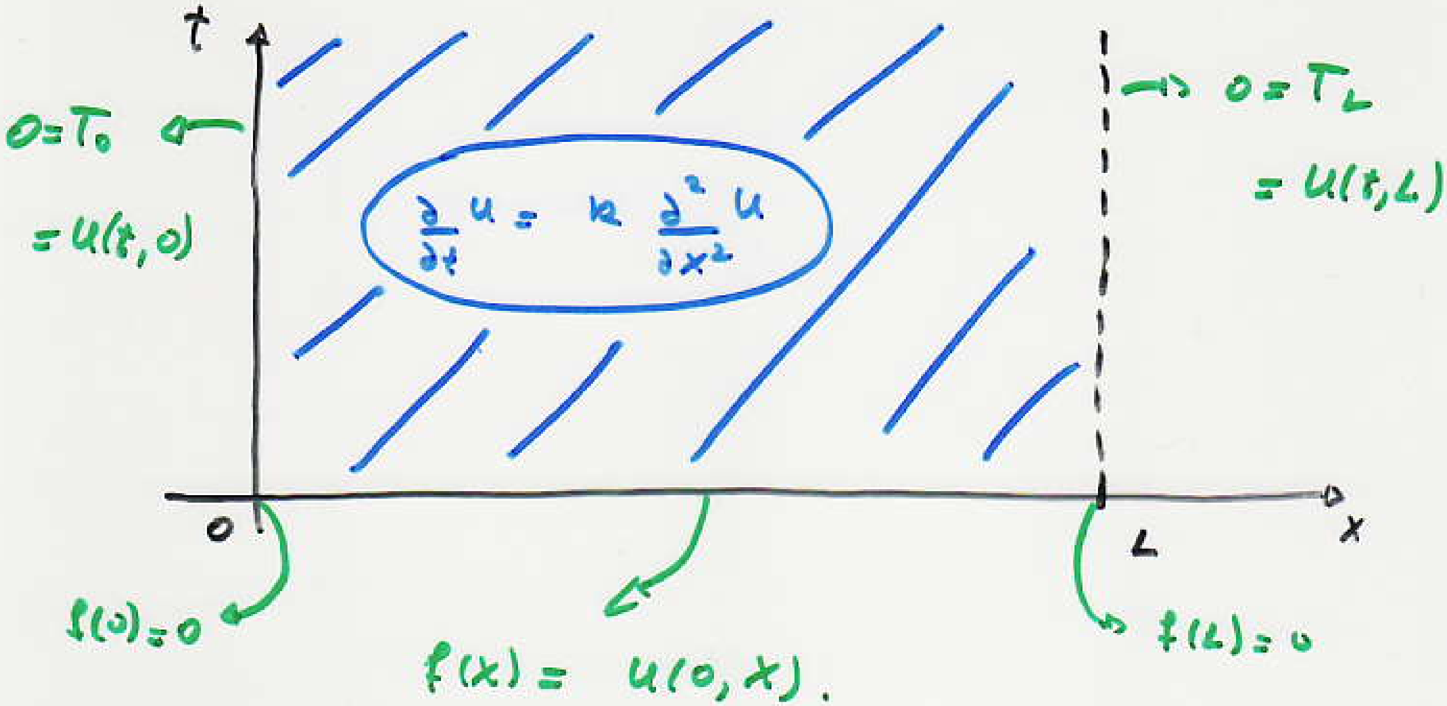
$$\boxed{\frac{\partial}{\partial t} u(t,x) = k \frac{\partial^2}{\partial x^2} u(t,x)} \qquad (4.1)$$

$$\boxed{u(0,x) = f(x)} \qquad \text{I.C.} \qquad (4.2)$$

$$\boxed{\begin{array}{l} u(t,0) = 0 \\ u(t,L) = 0 \end{array}} \qquad \text{B.C.} \qquad (4.3)$$

<u>Remark</u>: The b.c. in (4.3) are called homogeneous.

* The tx-diagram for the IBVP above is:



Domain of $u = [0,\infty) \times [0,L]$.

$$(t,x) \in [0,\infty) \times [0,L]$$

* <u>The separation of variables method</u>.

- The idea is to transform the PDE into infinitely many ODE.

- we describe this method in 6 steps.

Step 1.

One looks for solutions u given by an infinite series of simpler functions u_n, that is,

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x)} \tag{6.1}$$

where u_n is simpler than u in the following sense:

$$\boxed{u_n(t,x) = V_n(t) \quad W_n(x)} \tag{6.2}$$

Here c_n are constants, $n=1,\ldots$

Step 2.

Introduce (6.1) into the heat eq.

$$\frac{\partial}{\partial t} u - k \frac{\partial^2 u}{\partial x^2} = 0 \qquad (7.1).$$

$$\Updownarrow$$

$$\sum_{n=1}^{\infty} c_n \left[\frac{\partial}{\partial t} u_n - k \frac{\partial^2}{\partial x^2} u_n \right] = 0$$

A sufficient condition for (7.1) is

$$\boxed{\frac{\partial}{\partial t} u_n - k \frac{\partial^2}{\partial x^2} u_n = 0} \qquad n = 1, 2, \cdots \qquad (7.2)$$

Step 3.

Find u_n Sol. of the IBVP

$$\boxed{\frac{\partial}{\partial t} u_n(t,x) = k \frac{\partial^2}{\partial x^2} u_n(t,x)}$$

$$\boxed{u_n(0,x) = w_n(x)} \qquad \text{I.C.}$$

$$\boxed{\begin{aligned} u_n(t,0) &= 0 \\ u_n(t,L) &= 0 \end{aligned}} \qquad \text{B.C.}$$

with $\boxed{u_n(t,x) = v_n(t)\, w_n(x).}$

Step 4.

Transform the IBVP for u_n into:

(a) IVP for v_n.

(b) BVP for w_n.

(This is the key step.)

Notice:
$$\frac{\partial}{\partial t} u_n(t,x) = \frac{\partial}{\partial t}\left(v_n(t)\, w_n(x)\right)$$
$$= w_n(x)\, \frac{d}{dt} v_n(t)$$

$$\frac{\partial^2}{\partial x^2} u_n(t,x) = \frac{\partial^2}{\partial x^2}\left(v_n(t)\, w_n(x)\right)$$
$$= v_n(t)\, \frac{d^2}{dx^2} w_n(x).$$

$$W_n(x) \frac{d}{dt} V_n(t) = k \; V_n(t) \frac{d^2}{dx^2} W_n(x)$$

$$\boxed{\underbrace{\frac{1}{k} \frac{1}{V_n(t)} \frac{d}{dt} V_n(t)}_{\text{It depends only on } t} = \underbrace{\frac{1}{W_n(x)} \frac{d^2}{dx^2} W_n(x).}_{\text{It depends only on } x.}} \quad (10.1)$$

- The heat equation has the following property:

> The left-hand side of (10.1) depends only on t, while the right-hand side depends only on x.

- Whenever this happens in a PDE we can use the separation of variables method on that PDE.

From (10.1) we conclude that:

$$\boxed{\frac{1}{k} \frac{1}{V_n(t)} \frac{d}{dt} V_n(t) = -\lambda_n} \qquad (11.1)$$

$$\boxed{\frac{1}{W_n(x)} \frac{d^2 W_n(x)}{dx^2} = -\lambda_n} \qquad (11.2)$$

with λ_n: constant.

- So we transformed the original PDE (3.1) into infinitely many ODE (11.1) and (11.2), parametrized by n: positive integer.

- The original IBVP for the PDE (4.1)–(4.3) is transformed into:

(a) The IVP for v_n:

$$\left[\begin{array}{l} \dfrac{d}{dt} v_n(t) + k\,\lambda_n\, v_n(t) = 0 \\[2ex] \text{IC.} \quad v_n(0) = 1 \end{array}\right]$$

(b) The BVP for w_n:

$$\left[\begin{array}{l} \dfrac{d^2}{dx^2} w_n(x) + \lambda_n\, w_n(x) = 0 \\[2ex] \text{BC} \quad \left[\begin{array}{l} w_n(0) = 0 \\ w_n(L) = 0 \end{array}\right] \end{array}\right]$$

step 5.

(a1) Solve IVP (a) for V_n.

(b1) Solve BVP (b) for W_n.

Sol:

(a1)
$$\begin{bmatrix} V_n'(t) + k\lambda_n V_n(t) = 0 \\ V_n(0) = 1 \end{bmatrix}$$

Integrating factor method.

$$\mu(t) = e^{k\lambda_n t}$$

$$e^{k\lambda_n t} V_n' + k\lambda_n e^{k\lambda_n t} V_n = 0$$

$$\left(e^{k\lambda_n t} V_n(t) \right)' = 0$$

$$e^{k\lambda_n t} V_n(t) = c. \Rightarrow \boxed{V_n(t) = c e^{-\lambda_n k t}}$$

I.C. $1 = V_n(0) = c \Rightarrow \boxed{c = 1}$

$$\boxed{V_n(t) = e^{-k\lambda_n t}}$$

(b1) Eigenvalue - Eigenfunction Problem:

$$\left[\begin{array}{l} W_n''(x) + \lambda_n W_n(x) = 0 \\ \text{BC} \quad \left[\begin{array}{l} W_n(0) = 0 \\ W_n(L) = 0 \end{array}\right] \end{array}\right] \quad (14.1)$$

[We know that this problem has solutions only for $\lambda_n > 0$.]

Denote: $\boxed{\lambda_n = \mu_n^2}$

Proposing: $W_n(x) = e^{rx}$

$P(r) = r^2 + \mu_n^2 = 0 \Rightarrow \boxed{r = \pm \mu_n i}$

$$\boxed{W_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)}$$

(general sol. of (14.1).)

B.C. $\quad \left. \begin{array}{l} 0 = W_n(0) = c_1 \qquad \Rightarrow c_1 = 0 \\ 0 = W_n(L) = c_1 \cos(\mu_n L) + c_2 \sin(\mu_n L) \end{array} \right] \Rightarrow$

$$c_2 \sin(\mu_n L) = 0 \quad , \quad c_2 \neq 0 \quad \Rightarrow$$

$$\boxed{\sin(\mu_n L) = 0} \iff \boxed{\mu_n L = n\pi}$$

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2} \qquad \boxed{W_n(x) = \sin\left(\frac{n\pi}{L} x\right)}$$

$$\boxed{n = 1, 2, \dots}$$

We conclude:

$$\boxed{u_n(t,x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)} \qquad \boxed{n = 1, 2, \dots}$$

Step 6.

Compute the Solution to the IBVP for the heat eq. in (4.1)-(4.3).

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, u_n(t,x)$$

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)}$$

BC: $u(t,0) = 0$ ✓ $(\sin(0) = 0)$

$u(t,L) = 0$ ✓ $(\sin(n\pi) = 0)$

IC:

$$\boxed{f(x) = u(0,x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)} \quad (16.1)$$

we need to compute the coeffs. c_n.

- Recall the orthogonality relations:

$$\boxed{\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}}$$

Multiply (16.1) by $\sin\left(\frac{m\pi}{L}x\right)$ and integrate:

$$\sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi}{L}x\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

$$\frac{L}{2} c_m = \int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

$$\boxed{c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx .}$$

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right).}$$

-o-

Summary: IBVP for the heat eq.

Propose: $u(t,x) = \sum_{n=1}^{\infty} c_n \underbrace{u_n(t,x)}_{\text{simple.}}$

IBVP

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} c_n \, V_n(t) \, W_n(x)}$$

[Separation of variables method.]

[Fourier series coeffs.]

IVP

BVP

[e-val. e-function Problem.]

<u>Remark</u>: The separation of variables method does NOT work for every PDE.

<u>Example</u>: Determine whether the separation of variables method can be used in

$$\left(t + \frac{x}{c}\right) \frac{\partial^2 u(t,x)}{\partial t^2} + k \frac{\partial^2 u}{\partial x^2} = 0.$$

<u>Sol</u>.

Let $u(t,x) = V(t)\, W(x)$

so:

$$\boxed{\frac{\partial^2}{\partial t^2} u = W(x) \frac{d^2}{dt^2} V(t)}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = V(t) \frac{d^2}{dx^2} W(x)}$$

$$\left(t + \frac{x}{c}\right) W(x) \frac{d^2 V(t)}{dt^2} = -k\, V(t) \frac{d^2 W(x)}{dx^2}$$

$$\boxed{\underbrace{-\frac{1}{k}\left(t + \frac{x}{c}\right) \frac{1}{V(t)} \frac{d^2 V(t)}{dt^2}}_{\text{It depends on } t,\, x.} = \underbrace{\frac{1}{W(x)} \frac{d^2 W(x)}{dx^2}}_{\text{it depends on } x}}$$

$\boxed{\text{No}}$

Example Find the sol. to the IBVP

$$4 \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u \qquad x \in [0,2], \; t>0.$$

$$u(0,x) = 3 \sin\left(\frac{\pi x}{2}\right) \qquad \text{IC.}$$

$$u(t,0) = 0 \qquad u(t,2) = 0. \qquad \text{BC.}$$

Sol.

$$u_n(t,x) = v_n(t) \; w_n(x)$$

$$4 \; w_n(x) \frac{d}{dt} v_n(t) = v_n(t) \frac{d^2}{dx^2} w_n(x)$$

$$4 \frac{v_n'(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n$$

$$\boxed{v_n'(t) + \frac{\lambda_n}{4} v_n(t) = 0}$$

$$\boxed{w_n''(x) + \lambda_n w_n(x) = 0}$$

$$\boxed{V_n' + \frac{\lambda_n}{4} V_n = 0} \quad , \quad \boxed{V_n(0) = 1} \quad \Rightarrow$$

$$e^{\frac{\lambda_n}{4} t} V_n(t) = c$$

$$\Rightarrow \quad 1 = V(0) = c \quad \Rightarrow \quad c = 1$$

$$\boxed{V_n(t) = e^{-\frac{\lambda_n t}{4}}} \qquad \left(k = \frac{1}{4}\right)$$

$$\boxed{W_n'' + \lambda_n W_n = 0} \qquad \boxed{W_n(0) = W_n(L) = 0}$$

$$\lambda_n > 0 \quad \Rightarrow \quad \boxed{\lambda_n = \pm \mu_n i}$$

$$\boxed{W_n(x) = c_1 \cos(\mu_n x) + c_2 \sin(\mu_n x)}$$

$$W_n(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

$$W_n(2) = 0 \quad \Rightarrow \quad \sin(\mu_n 2) = 0 \quad \Leftrightarrow \quad \mu_n 2 = n\pi$$

$$\boxed{\lambda_n = \left(\frac{n\pi}{2}\right)^2} \qquad \boxed{W_n(x) = \sin\left(\frac{n\pi}{2} x\right)}$$

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-\left(\frac{n\pi}{2}\right)^2 \frac{t}{4}} \sin\left(\frac{n\pi}{2}x\right)}$$

I.C.

$$3 \sin\left(\frac{\pi}{2}x\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{2}x\right)$$

orthogonality of sine functions implies.

$$3 \int_0^2 \sin\left(\frac{\pi}{2}x\right) \sin\left(\frac{m\pi}{2}x\right) dx = \sum_{n=1}^{\infty} c_n \int_0^2 \sin\left(\frac{n\pi}{2}x\right) \sin\left(\frac{m\pi}{2}x\right) dx$$

$m \neq 1$

$$0 = c_m \left(\frac{2}{2}\right) \Rightarrow \boxed{c_m = 0 \quad m \neq 1}$$

$$\boxed{3 \sin\left(\frac{\pi}{2}x\right) = c_1 \sin\left(\frac{\pi}{2}x\right)} \Rightarrow \boxed{c_1 = 3}$$

So:

$$\boxed{u(t,x) = 3 \, e^{-\frac{n^2\pi^2}{16}t} \sin\left(\frac{\pi}{2}x\right)}$$