mth 235 L39

Plan:

* Even, odd functions
* Main properties
* Sine series, cosine series.
* Even-periodic, odd-periodic extensions of functions.

(10.4)

Common Final Exam:

Tuesday May 4, 10:00 am - 12:00 pm.

Place: Sections 09, 12 :
N 130 Business College complex.

Sections 10, 11 :
1345 Engineering Building (here!)

Common makeup exam: Wednesday May 5.

Place : 147 Communications Art Building.

* Even, odd functions

Def: A function $f : [-L, L] \to \mathbb{R}$ is called even iff

$$\boxed{f(-x) = f(x)} \; ;$$

and f is called odd iff

$$\boxed{f(-x) = -f(x).}$$

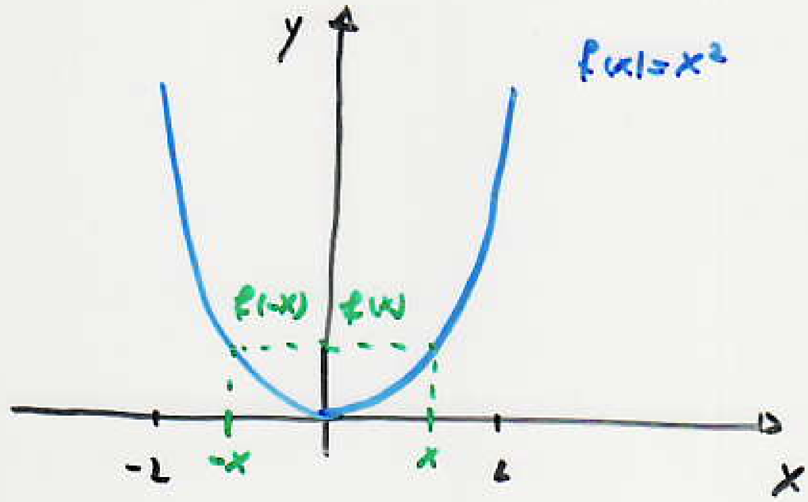
Remarks:

- The only function that is both odd and even is $f = 0$.
- Most functions are neither odd nor even.

* Examples

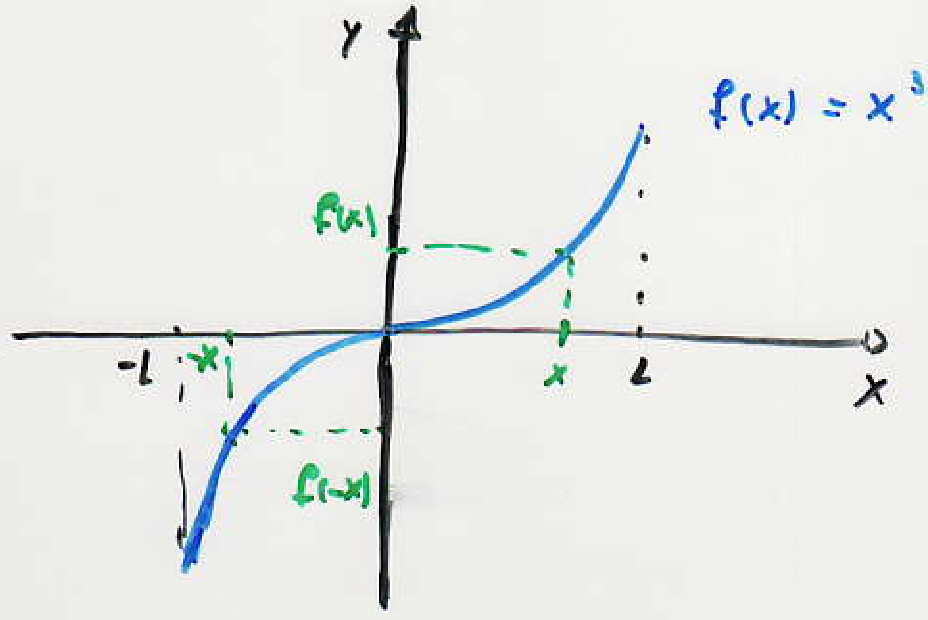
(1) $f(x) = x^2$ is even on $[-L, L]$, since

$$f(-x) = (-x)^2 = x^2 = f(x)$$



(2) $f(x) = x^3$ is odd on $[-L, L]$, since

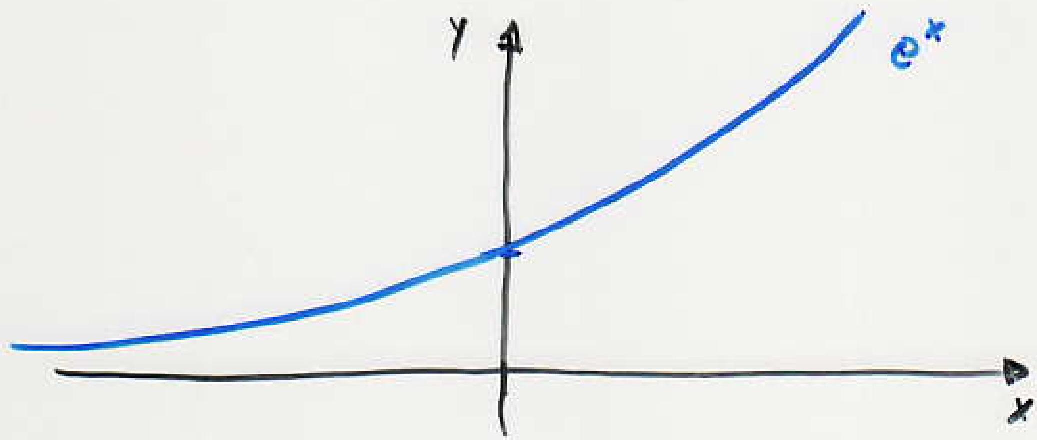
$$f(-x) = (-x)^3 = -x^3 = -f(x).$$



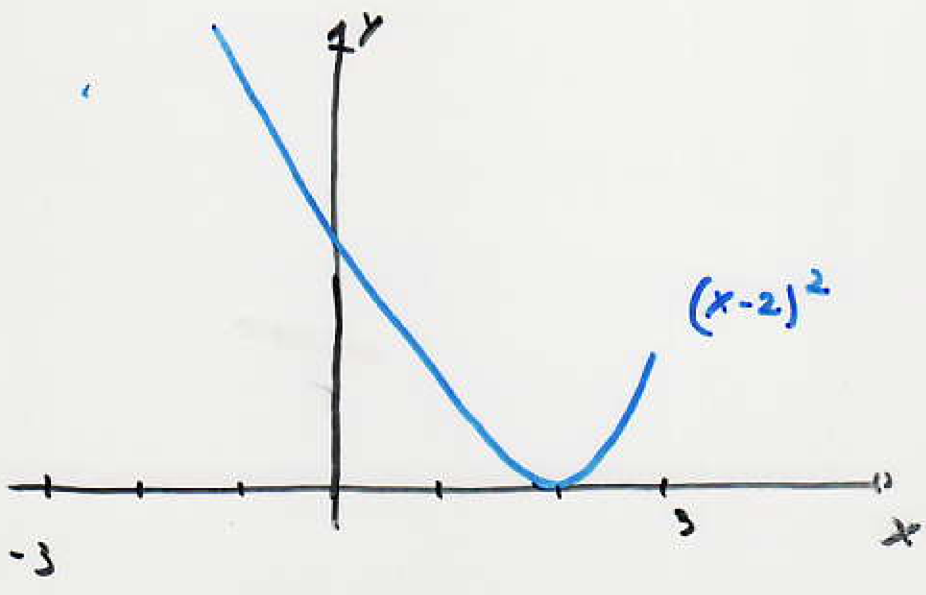
(3) $f(x) = \cos(x)$ is even on $[-L, L]$.

(4) $f(x) = \sin(x)$ is odd on $[-L, L]$.

(5) $f(x) = e^x$ is neither even nor odd



(6) $f(x) = (x-2)^2$ is neither even nor odd



* Properties of even, odd functions.

Propos.: (1) A linear combination of even (odd) function is even (odd).

(2) The product of two odd functions is even

(3) The product of two even functions is even.

(4) The product of an even function by an odd function is odd.

Proof.

(1) f, g even, that is, $f(-x) = f(x)$, $g(-x) = g(x)$,

then, for $a, b \in \mathbb{R}$ holds

$$(af + bg)(-x) = a\, f(-x) + b\, g(-x)$$

$$= a\, f(x) + b\, g(x)$$

$$= (af + bg)(x) \quad \checkmark$$

case "odd" similar.

(2) f, g odd, that is, $f(-x) = -f(x)$
$g(-x) = -g(x)$.

then, $(fg)(-x) = f(-x)\, g(-x)$

$= [-f(x)]\, [-g(x)]$

$= f(x)\, g(x)$

$= (fg)(x)$.

cases (3), (4) similar $\blacksquare$

Propos. If $f : [-L, L] \to \mathbb{R}$ is even, then

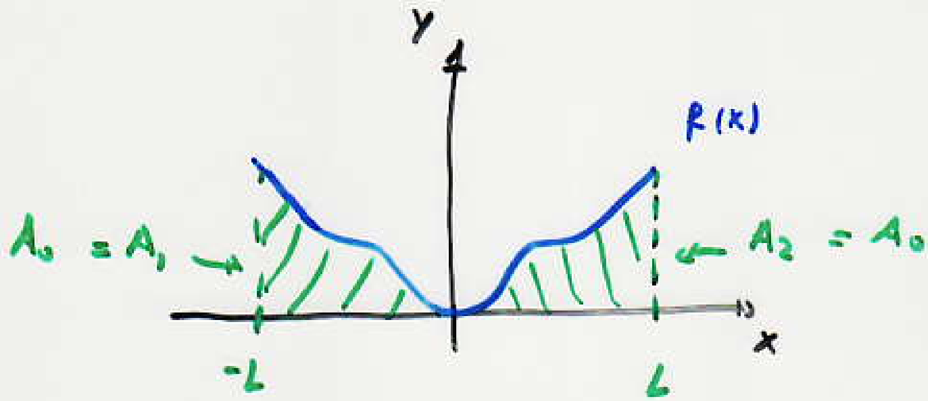
$$\boxed{\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.}$$

If $f : [-L, L] \to \mathbb{R}$ is odd, then

$$\boxed{\int_{-L}^{L} f(x)\,dx = 0}$$

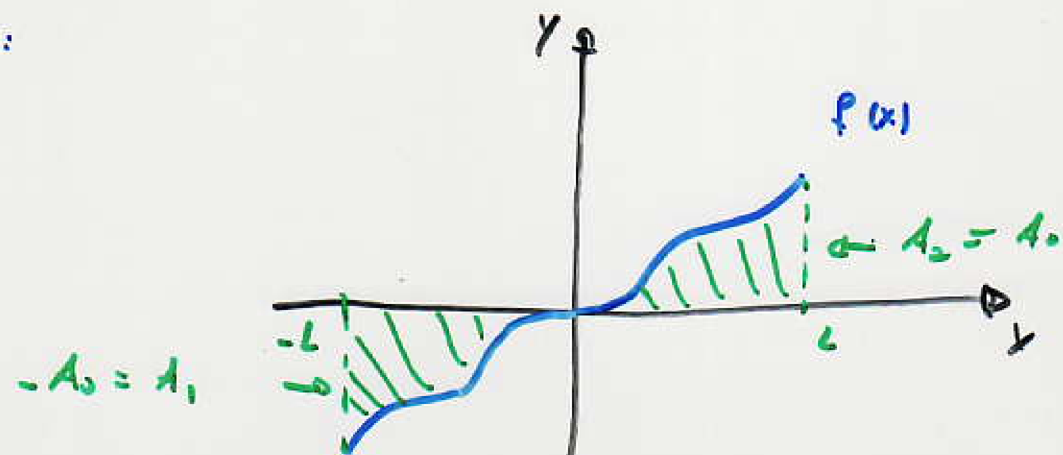
__Examples__

Even case :



$$A = A_1 + A_2 = 2A_0$$

Odd case :



$$A_1 + A_2 = 0.$$

<u>Proof</u> : $\displaystyle\int_{-L}^{L} f(x)\,dx = \underbrace{\int_{-L}^{0} f(x)\,dx}_{y=-x,\ dy=-dx} + \int_{0}^{L} f(x)\,dx$

$$= \int_{L}^{0} f(-y)\,(-dy) + \int_{0}^{L} f(x)\,dx$$

$$= \int_{0}^{L} f(-y)\,dy + \int_{0}^{L} f(x)\,dx$$

even case : $f(-y) = f(y)$

$$\boxed{\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx}$$

odd case : $f(-y) = -f(y)$

$$\boxed{\int_{-L}^{L} f(x)\,dx = 0}.$$

∎

* Sine series, cosine series.

Thm: consider the function $f:[-L,L]\to\mathbb{R}$ with Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right].$$

(1) If f is even, then $b_n = 0$, $n=1,\dots$ and the Fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}$$

is called a cosine series.

(2) If f is odd, then $a_n = 0$, $n=0,1,\dots$ and the Fourier series

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}$$

is called a sine series.

Proof: (1) f even:

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{even}}{\underset{\uparrow}{f(x)}} \quad \underset{\text{odd}}{\underset{\uparrow}{\sin\left(\frac{n\pi x}{L}\right)}}}_{\text{odd}} \, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} (\text{odd function})\, dx \implies \boxed{b_n = 0}$$

(2) f odd:

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{odd}}{\underset{\uparrow}{f(x)}} \quad \underset{\text{even}}{\underset{\uparrow}{\cos\left(\frac{n\pi x}{L}\right)}}}_{\text{odd}} \, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} (\text{odd function})\, dx \implies \boxed{a_n = 0}$$

$\blacksquare$

* Even-periodic, odd-periodic extensions of a function.

(1) even-periodic case:

A function $f: [0, L] \to \mathbb{R}$ can be extended as an even function $f: [-L, L] \to \mathbb{R}$ requiring

$$\boxed{f(-x) = f(x) \qquad \text{for} \quad x \in [0, L].}$$

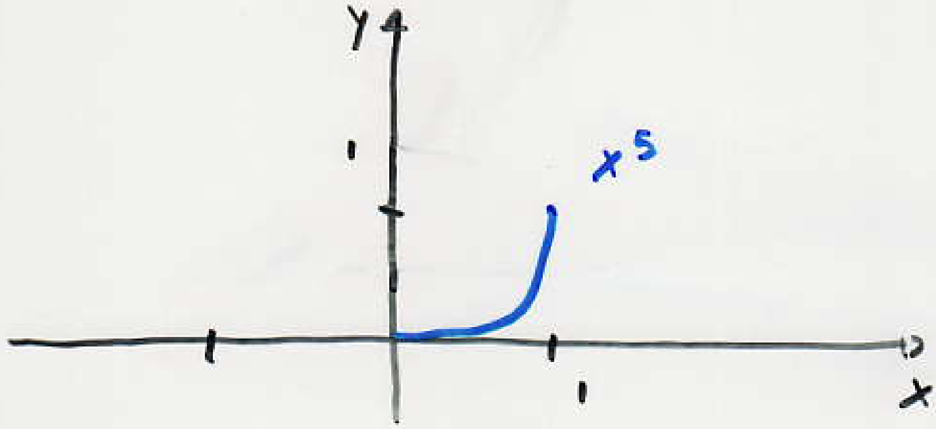
This $f: [-L, L] \to \mathbb{R}$ can be extended as a periodic function $f: \mathbb{R} \to \mathbb{R}$ requiring

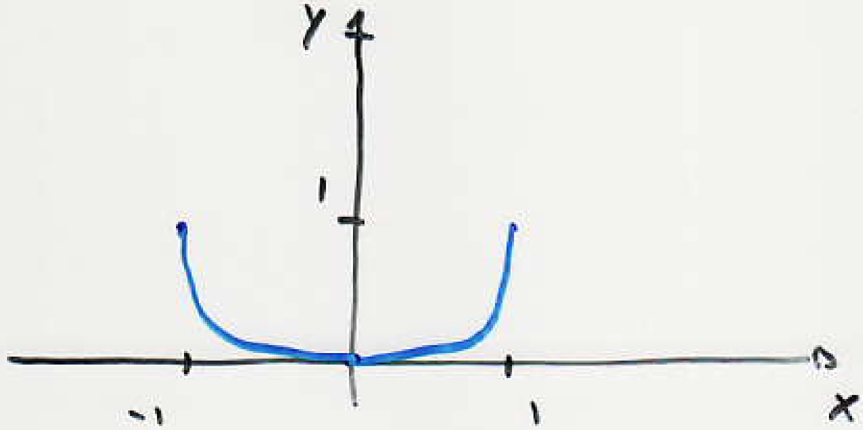
$$\boxed{f(x + 2nL) = f(x) \qquad x \in [-L, L], \; n \text{ : integer.}}$$

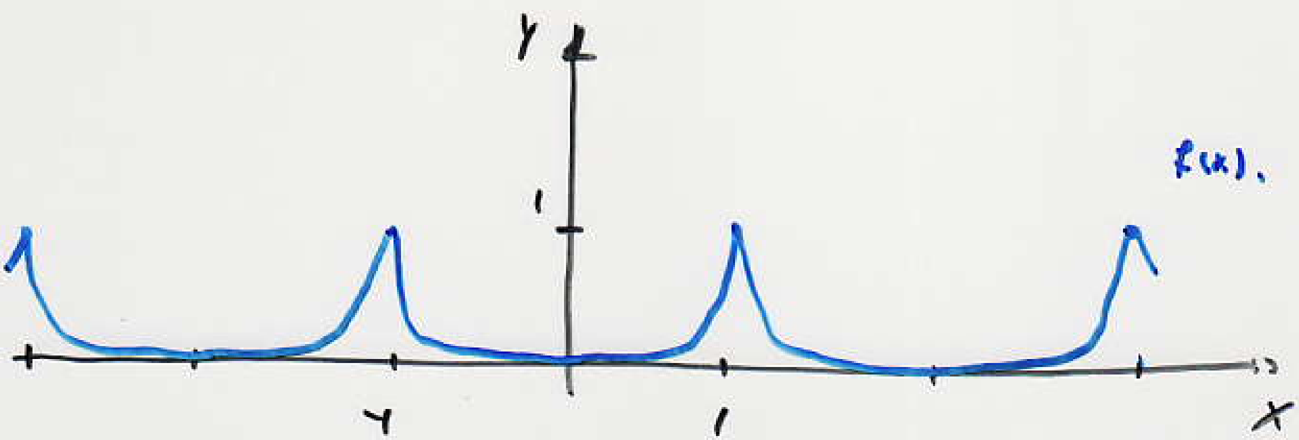
Example: Sketch the graph of the even-periodic extension of

$$f(x) = x^5 \qquad x \in [0,1].$$

Sol:







(2) odd - periodic case.

A function $f : [0, L) \to \mathbb{R}$ can be extended as an odd periodic function $f : (-L, L) \to \mathbb{R}$ requiring

$$\boxed{f(-x) = -f(x) \qquad \text{for} \quad x \in [0, L).}$$

This $f : (-L, L) \to \mathbb{R}$ can be extended as a periodic function $f : \mathbb{R} \to \mathbb{R}$ requiring:

$$\boxed{\begin{aligned} &f(x + 2nL) = f(x) && x \in (-L, L) \\ &f(nL) = 0 && n : \text{integer.} \end{aligned}}$$

Remark. At $x = \pm L$, function f satisfies:

(a) f is odd $\Rightarrow$ $\boxed{f(-L) = -f(L)}$

(b) f is periodic $\Rightarrow$ $f(-L) = f(-L + 2L)$

$$= f(L)$$

Therefore: $-f(L) = f(L)$

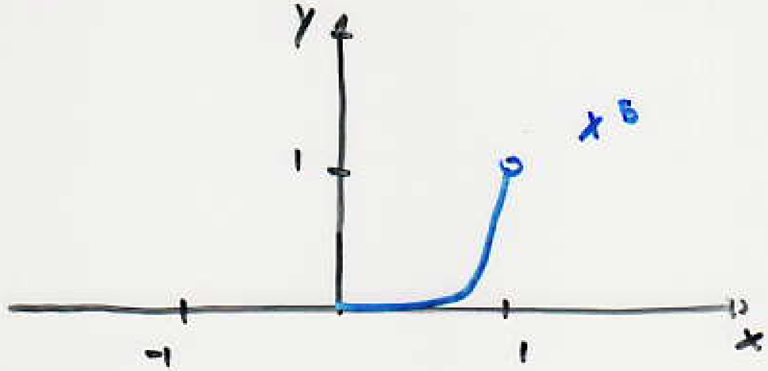
$$\boxed{f(L) = 0}$$

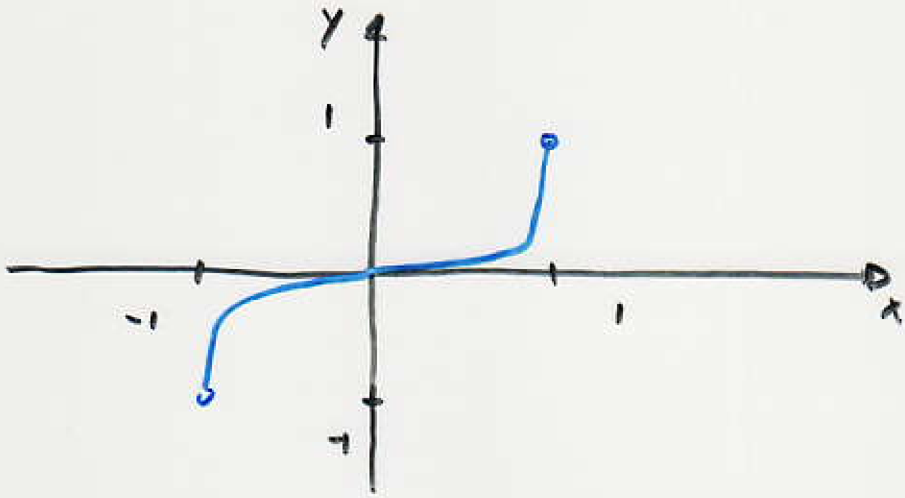
* <u>Example</u>

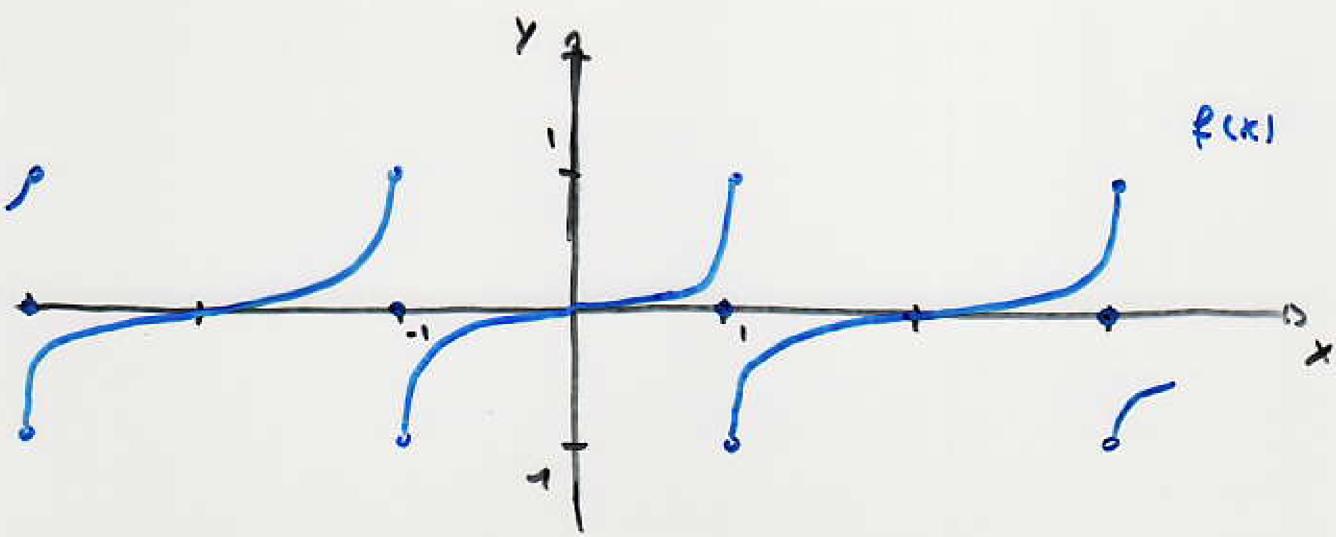
Sketch the graph of the odd-periodic extension of

$$f(x) = x^5 \quad , \qquad x \in [0, 1) .$$

<u>Sol</u>:





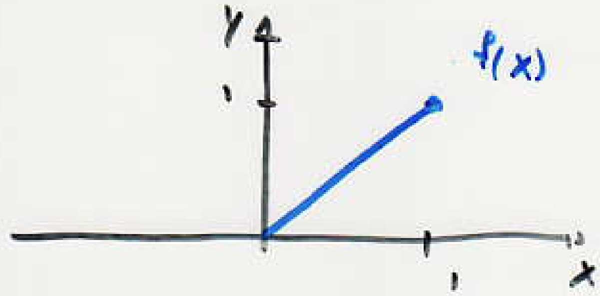


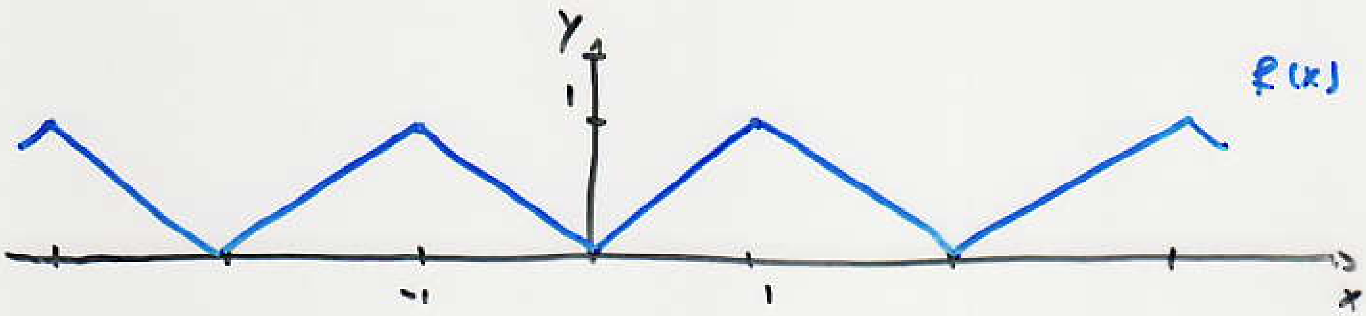
* <u>Example</u> [Sketch the graph of the even-periodic extension of

$$f(x) = x, \qquad x \in [0,1]$$

and then find its Fourier series.]

<u>Sol:</u>





The Fourier series of f, even-periodic, satisfies:

$$\boxed{b_n = 0} \qquad n = 1, \dots$$

From graph: $\boxed{a_0 = 1}$.

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\text{even}}{f(x)} \quad \underset{\text{even}}{\cos\left(\frac{n\pi x}{L}\right)}}_{\text{even}}\, dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x) \quad \cos\left(\frac{n\pi x}{L}\right) dx$$

$$a_n = 2\int_{0}^{1} x \quad \cos(n\pi x) \quad dx$$ integr. by parts

$$= 2\left[\underbrace{\frac{x \sin(n\pi x)}{n\pi}}_{=0} + \frac{\cos(n\pi x)}{(n\pi)^2} \right]\Bigg|_0^1$$

$$= \frac{2}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

$$\boxed{a_n = \frac{2}{(n\pi)^2}\left[(-1)^n - 1\right]}$$

$n = 2k \Rightarrow \qquad a_{2k} = \frac{2}{(2k)^2\pi^2}\left[(-1)^{2k} - 1\right]$

$$\boxed{a_{2k} = 0}$$

$n = 2k+1 \Rightarrow \qquad a_{2k+1} = \frac{2}{(2k+1)^2\pi^2}\left[(-1)^{2k+1} - 1\right]$

$$\boxed{a_{2k+1} = \frac{-4}{(2k+1)^2\pi^2}}$$

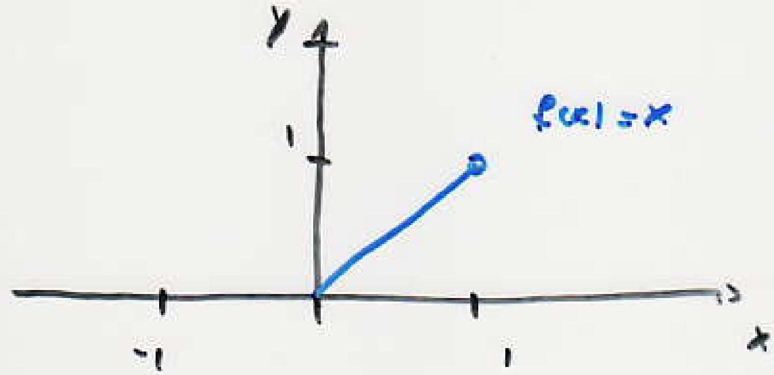
$$\boxed{f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\left((2k+1)\pi x\right)}$$

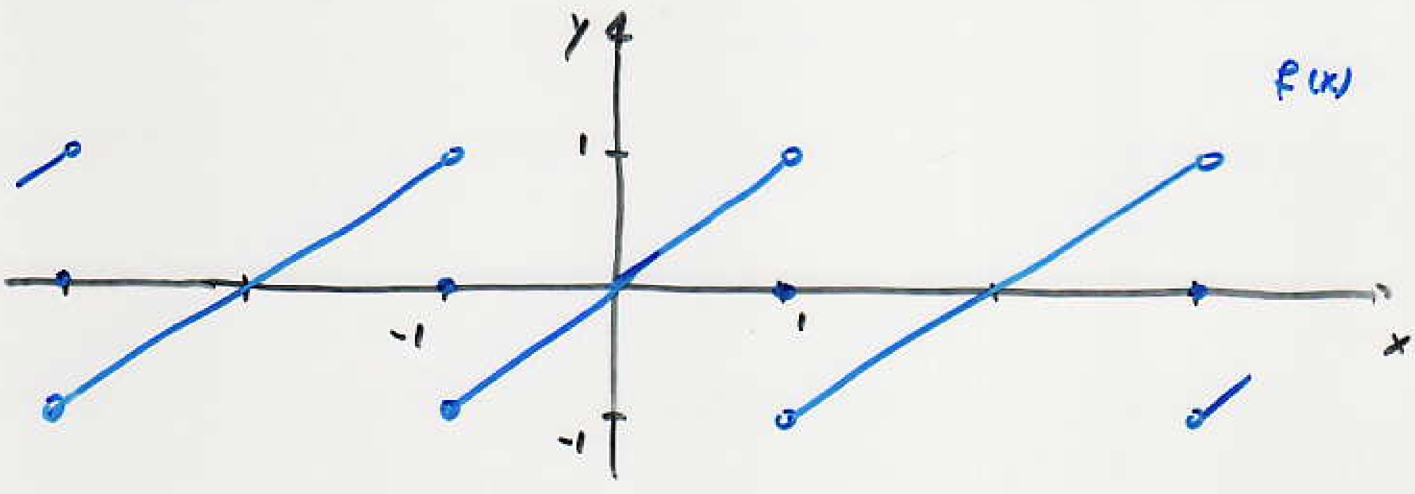
<u>Example</u> [Sketch the graph of the odd-periodic extension of

$$f(x) = x \,, \qquad x \in [0, 1)$$

and find its Fourier series.]

<u>Sol.</u>





The Fourier series of an odd-periodic function satisfies

$$\boxed{a_n = 0} \qquad n = 0, 1, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\substack{\uparrow \\ \text{odd}}}{f(x)} \quad \underset{\substack{\uparrow \\ \text{odd}}}{\sin\left(\frac{n\pi x}{L}\right)}}_{\text{even.}} \, dx$$

$$b_n = \frac{2}{L}\int_0^L f(x) \, \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = 2\int_0^1 x \, \sin(n\pi x)\, dx$$

integr. by parts.

$$b_n = 2\left[-\frac{x\cos(n\pi x)}{n\pi} + \underbrace{\frac{\sin(n\pi x)}{(n\pi)^2}}_{=0} \right]\Bigg|_0^1$$

$$b_n = -\frac{2}{n\pi}\left[\cos(n\pi) - 0\right]$$

$$b_n = -\frac{2}{n\pi}(-1)^n \Rightarrow \boxed{b_n = \frac{2}{n\pi}(-1)^{n+1}}$$

$$\boxed{f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x)}$$