mth 235 L38

Plan:
* Fourier Thrm: continuous case
* Example.
* Fourier Thrm: piecewise continuous case
* Example.

(10.3)

* Fourier Thrm : Continuous case

Thrm : If function $f : [-L, L] \to \mathbb{R}$ is continuous,
then f can be expressed as

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$

with a_n, b_n given by

$$\boxed{a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$

$$\boxed{b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

Furthermore, $f : [-L, L] \to \mathbb{R}$ can be extended as a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $2L$.

Sketch of the proof:

- Define the partial sum functions

$$\boxed{f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$

With a_n, b_n given by

$$\left[\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned} \right]$$

- Express f_N as a convolution of sine, cosine functions and f.

- Use that convolution to show that

$$f_N(x) \to f(x)$$

$$N \to \infty.$$

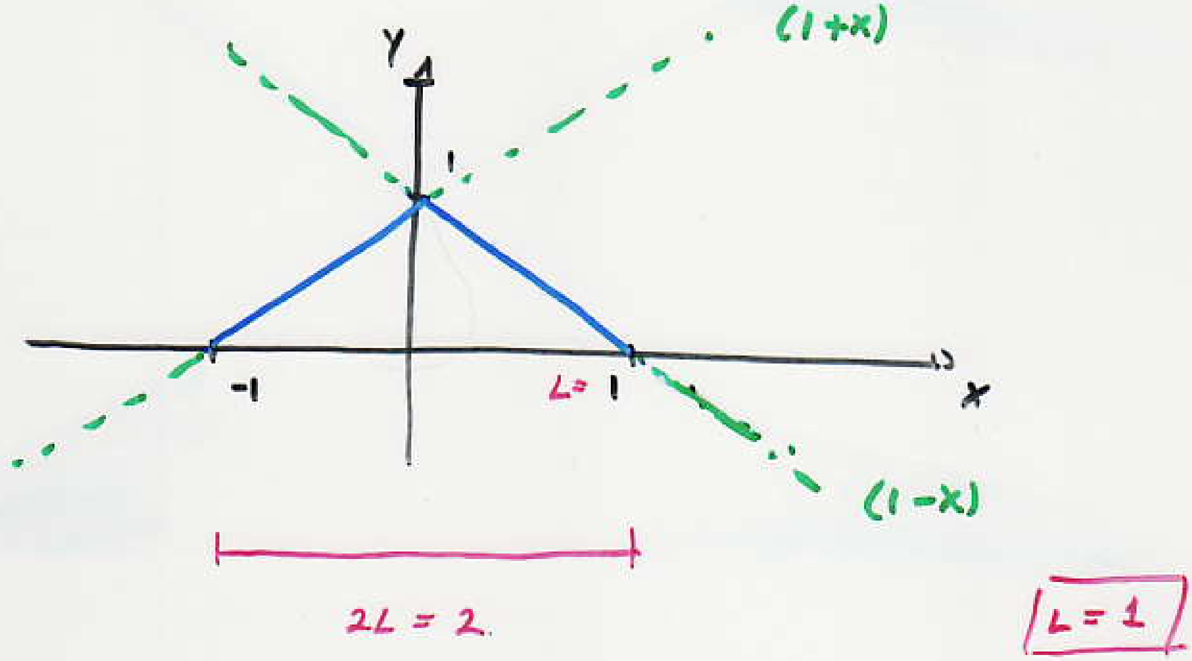
—o—

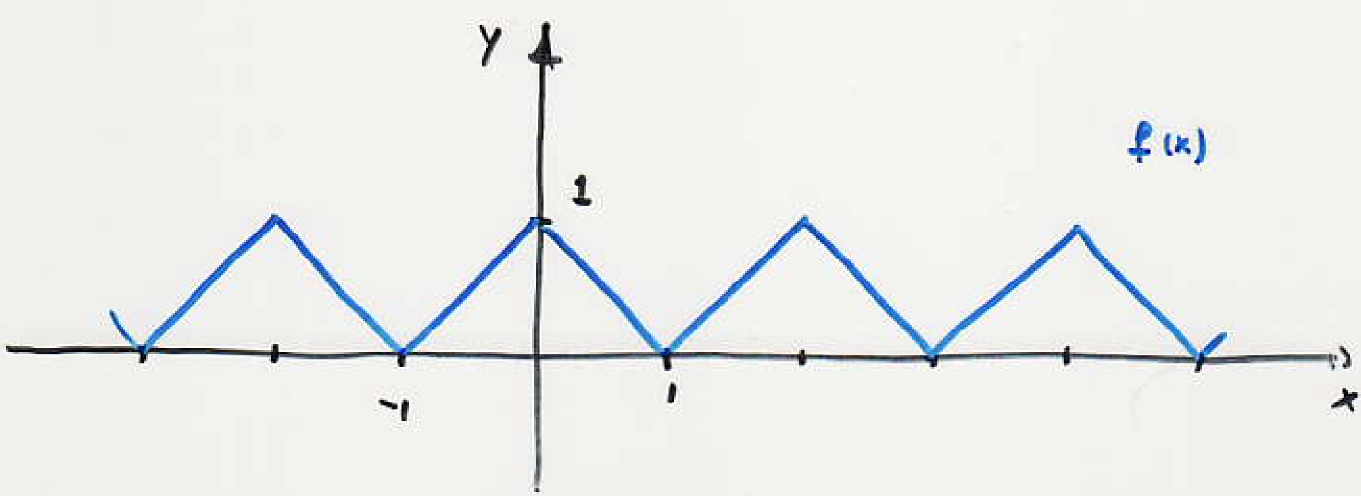
Example: Find the Fourier expansion of

$$f(x) = \begin{cases} 1+x & x \in [-1, 0] \\ 1-x & x \in [0, 1] \end{cases}$$

and 2-periodic on $\mathbb{R}$.

Sol.





We have to express $f(x)$ as:

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x) \right]}$$

Where:

$$a_0 = \frac{1}{1} \int_{-1}^{1} f(x) \, dx \qquad (\cos(0x) = 1)$$

$$= \int_{-1}^{0} (1+x) \, dx + \int_{0}^{1} (1-x) \, dx$$

$$= \left(x + \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(x - \frac{x^2}{2}\right)\Big|_{0}^{1}$$

$$= \left[0 - \left(-1 + \frac{1}{2}\right)\right] + \left[\left(1 - \frac{1}{2}\right) - 0\right]$$

$$= 1 - \frac{1}{2} + 1 - \frac{1}{2}$$

$$= 2 - 1$$

$$= 1 \Rightarrow \boxed{a_0 = 1}$$

$$\boxed{a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx}$$

$$a_n = \int_{-1}^{0} (1+x)\cos(n\pi x)\,dx$$

$$+ \int_{0}^{1} (1-x)\cos(n\pi x)\,dx$$

$$\boxed{\begin{aligned} a_n = & \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{-1}^{0} x\cos(n\pi x)\,dx \\ & + \int_{0}^{1} \cos(n\pi x)\,dx - \int_{0}^{1} x\cos(n\pi x)\,dx \end{aligned}}$$

$$\boxed{\int \cos(n\pi x)\,dx = \frac{1}{n\pi}\sin(n\pi x)}$$

Indefinite integrals

$$\boxed{\int x\cos(n\pi x)\,dx = \frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)}$$

integration by parts.

$u = x \qquad v' = \cos(n\pi x)$

$u' = 1 \qquad v = \frac{1}{n\pi}\sin(n\pi x)$

$$\left[\begin{aligned} a_n = {} & \frac{1}{n\pi} \underbrace{\sin(n\pi x)}_{=0}\Big|_{-1}^{0} \\ & + \left[\frac{x}{n\pi} \underbrace{\sin(n\pi x)}_{=0} + \frac{1}{(n\pi)^2} \cos(n\pi x) \right]\Big|_{-1}^{0} \\ & + \frac{1}{n\pi} \underbrace{\sin(n\pi x)}_{=0}\Big|_{0}^{1} \\ & - \left[\frac{x}{n\pi} \underbrace{\sin(n\pi x)}_{=0} + \frac{1}{(n\pi)^2} \cos(n\pi x) \right]\Big|_{0}^{1} \end{aligned}\right]$$

$$a_n = \frac{1}{(n\pi)^2}\left(1 - \cos(-n\pi)\right) - \frac{1}{(n\pi)^2}\left(\cos(n\pi) - 1\right)$$

$$a_n = \frac{1}{(n\pi)^2}\left[1 - \cos(n\pi) - \cos(n\pi) + 1 \right]$$

$$\boxed{a_n = \frac{2}{n^2\pi^2}\left(1 - \cos(n\pi)\right)}$$

Recall: $\boxed{\cos(n\pi) = (-1)^n}$

$$\boxed{a_n = \frac{2}{n^2\pi^2}\left[1+(-1)^{n+1}\right]}$$

<u>Exercise</u>: Show: $\boxed{b_n = 0}$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}\left(1+(-1)^{n+1}\right)\cos(n\pi x)}$$

If: $n+1 = 2k+1 \Rightarrow a_{2k+1} = \frac{2}{(2k+1)^2\pi^2}(1-1)$

$$\boxed{a_{2k+1} = 0}$$

If: $n+1 = 2k \Rightarrow a_{2k} = \frac{2}{(2k)^2\pi^2}(1+1)$

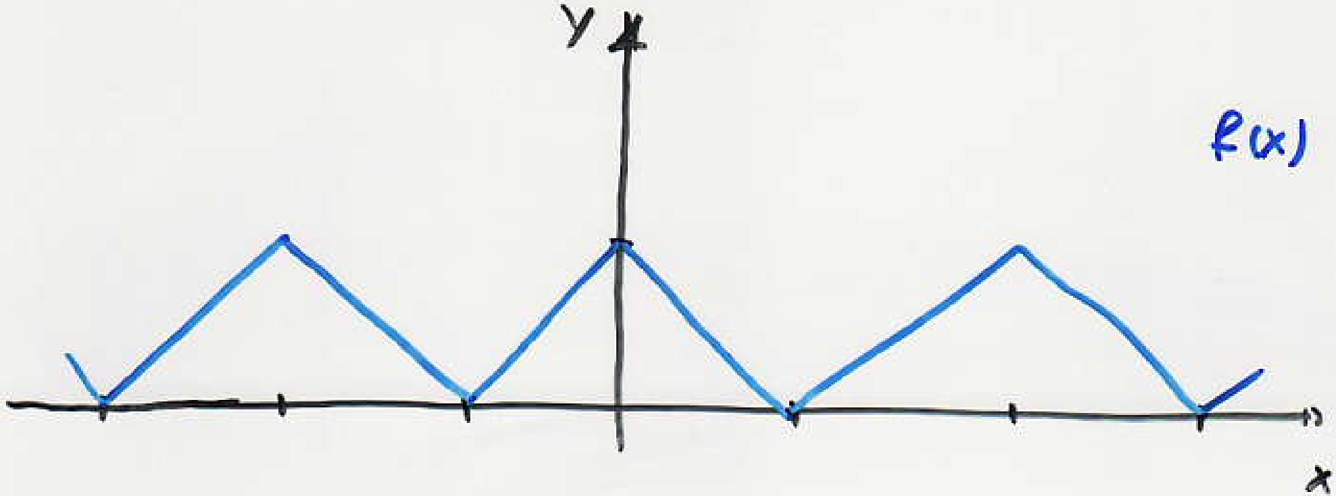
$\downarrow$

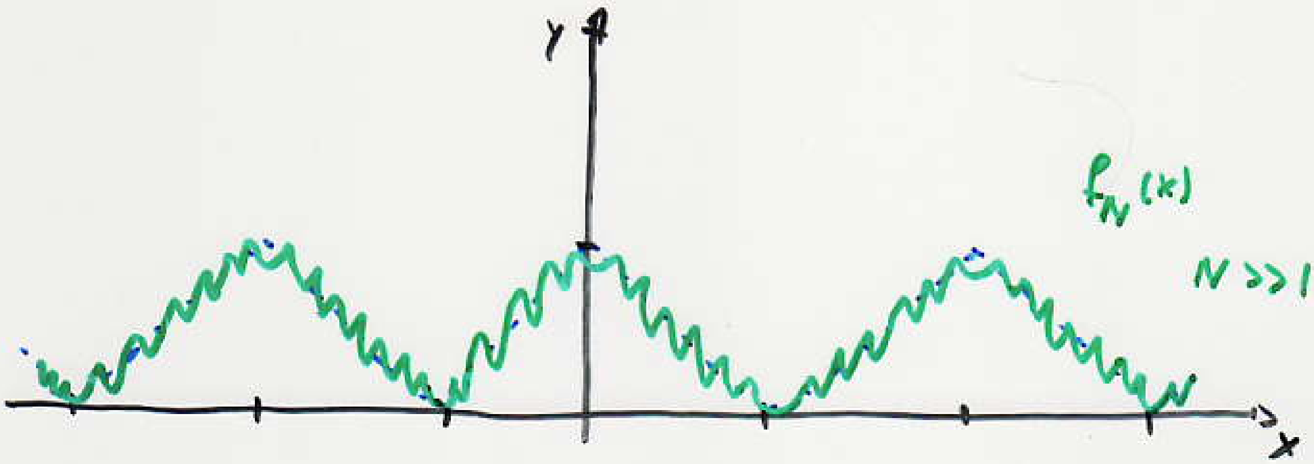
$n = 2k-1$

$$\boxed{a_{2k} = \frac{4}{4k^2\pi^2}}$$

$$\boxed{f(x) = \frac{1}{2} + \frac{1}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{k^2}\cos((2k-1)\pi x)}$$

$$f_N(x) = \frac{1}{2} + \frac{1}{\pi^2} \sum_{k=1}^{N} \frac{1}{k^2} \cos\big((2k-1)\pi x\big).$$





* **The Fourier Thrm: Piecewise continuous case**

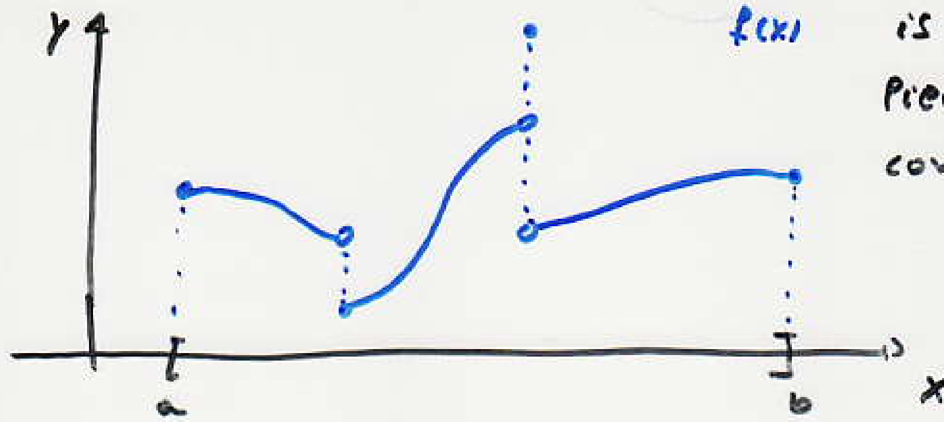
Def: A function $f : [a,b] \to \mathbb{R}$ is called piecewise continuous iff it holds:

(1) $[a,b]$ can be partitioned in a finite number of subintervals such that f is continuous on the interior of these subintervals.

(2) f has finite limits at the endpoints of all subintervals.

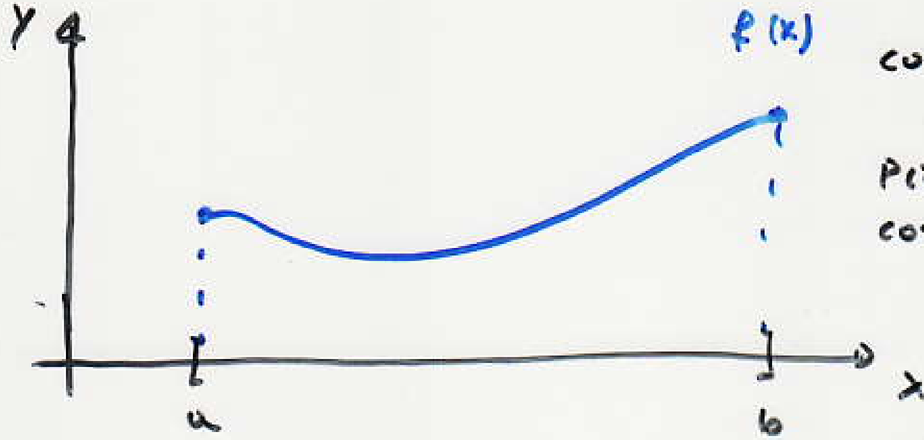
* <u>Examples</u>

(1)



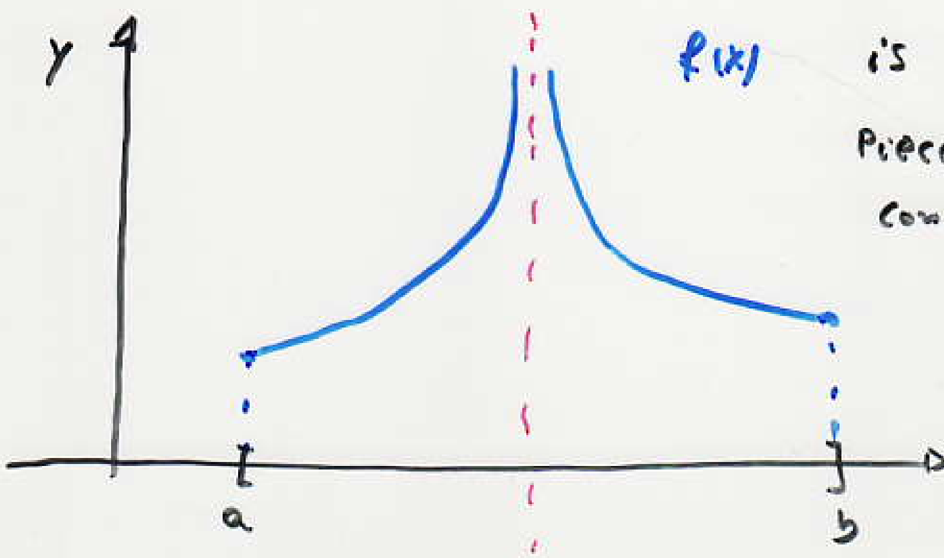
f(x) is piecewise continuous.

(2)



f(x) continuous is piecewise continuous.

(3)



f(x) is NOT piecewise continuous

Thrm: If $f : [-L, L] \to \mathbb{R}$ is piecewise continuous, Then the function

$$\boxed{f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$

where a_n, b_n are given by

$$\boxed{a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$

$$\boxed{b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

satisfies that:

(1) $f_F(x) = f(x)$ for all x where f is continuous,

(2) $$\boxed{f_N(x_0) = \frac{1}{2} \left[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x) \right]}$$

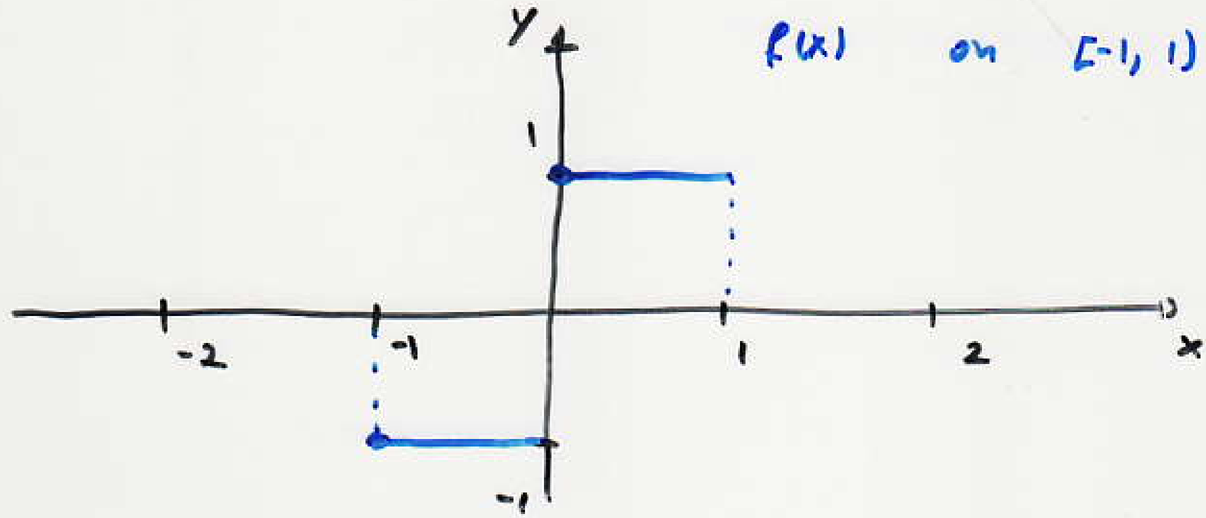
for all x_0 where f is discontinuous.

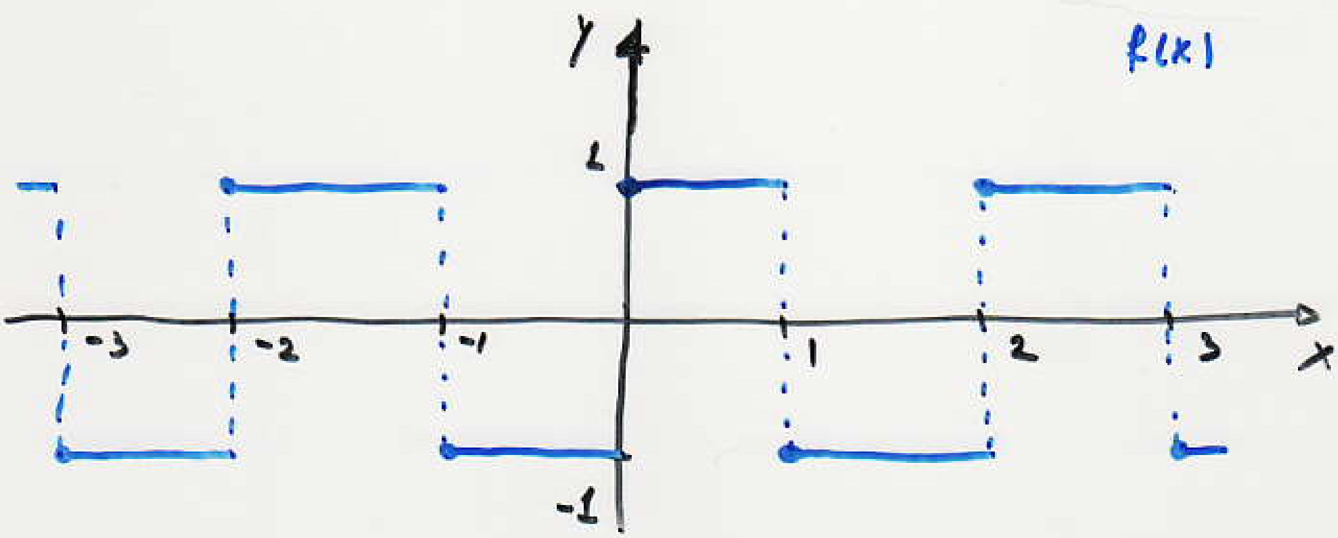
* **Example** [Find the Fourier Series of

$$f(x) = \begin{cases} -1 & x \in [-1, 0) \\ 1 & x \in [0, 1) \end{cases}$$

and periodic with period 2.]

Sol.





$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx \quad , \quad \boxed{L = 1}$$

$$b_n = \int_{-1}^{0}(-1)\,\sin(n\pi x)\,dx + \int_{0}^{1}(1)\,\sin(n\pi x)\,dx$$

$$b_n = \frac{(-1)}{n\pi}\left[-\cos(n\pi x)\Big|_{-1}^{0}\right] + \frac{1}{n\pi}\left[-\cos(n\pi x)\Big|_{0}^{1}\right]$$

$$b_n = \frac{(-1)}{n\pi}\left[-1 + \cos(-n\pi)\right] + \frac{1}{n\pi}\left[-\cos(n\pi) + 1\right]$$

$$b_n = \frac{1}{n\pi}\left[1 - \cos(n\pi) - \cos(n\pi) + 1\right]$$

$$\boxed{b_n = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]}$$

$$\boxed{\cos(n\pi) = (-1)^n}$$

$$\boxed{b_n = \frac{2}{n\pi}\left[1 - (-1)^n\right]}$$

If $n = 2k$, Then $b_{2k} = \frac{2}{2k\pi}\left(1 - (-1)^{2k}\right)$

$$\boxed{b_{2k} = 0}$$

If $n = 2k+1$, Then $b_{2k+1} = \frac{2}{(2k+1)\pi}\left(1 - (-1)^{2k+1}\right)$

$$\boxed{b_{2k+1} = \frac{4}{(2k+1)\pi}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad , \quad \boxed{L = 1}$$

$$a_n = \int_{-1}^{0} (-1)\cos(n\pi x)\,dx + \int_{0}^{1} (1)\cos(n\pi x)\,dx$$

$$a_n = \frac{(-1)}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} + \frac{1}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

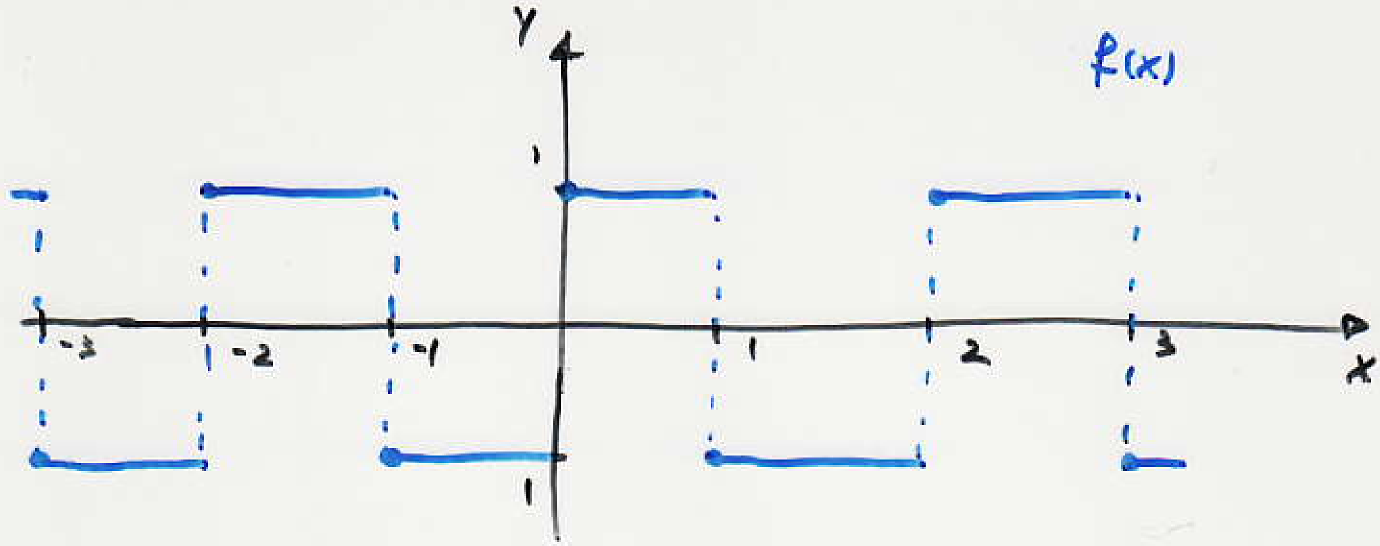
$$a_n = \frac{(-1)}{n\pi}\left[0 - \sin(-n\pi)\right] + \frac{1}{n\pi}\left[\sin(n\pi) - 0\right]$$

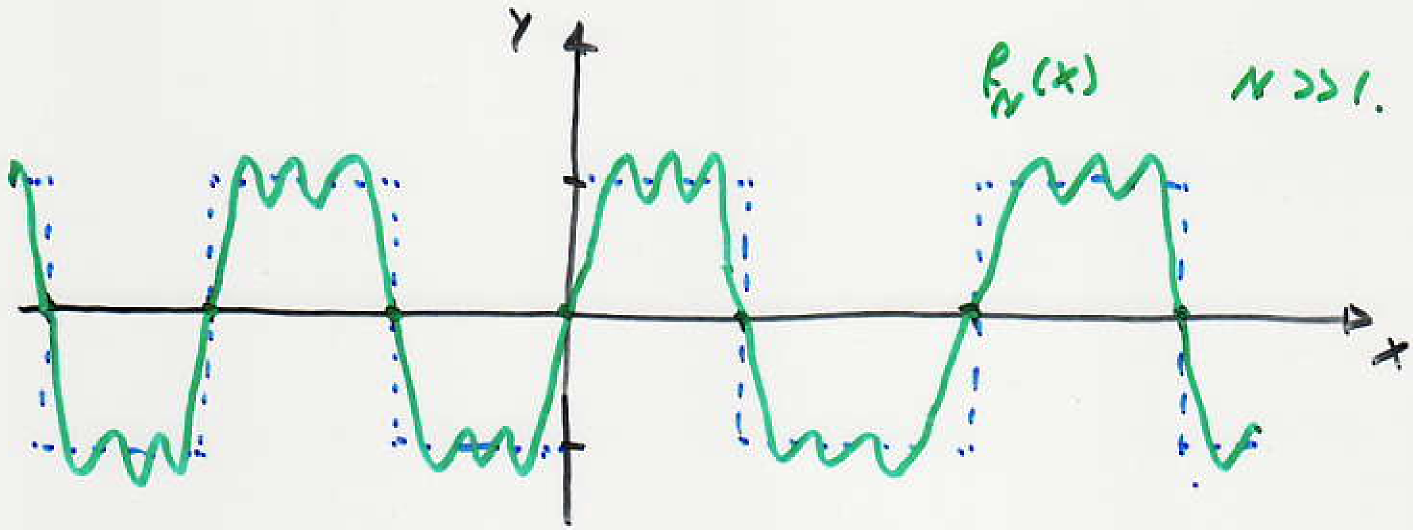
$$a_n = \frac{1}{n\pi}\left[-\sin(n\pi) + \sin(n\pi)\right]$$

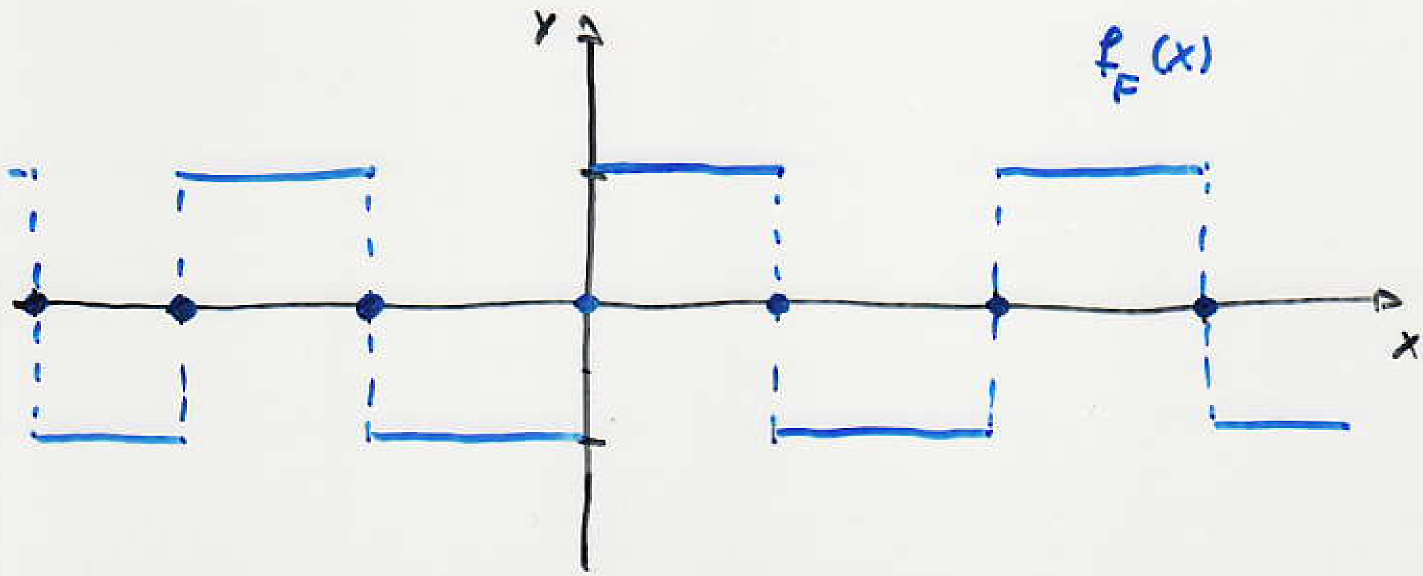
$$\boxed{a_n = 0}$$

$$\boxed{f_F(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin\left((2k+1)\pi x\right)}$$

$$\boxed{f_F(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin\left((2k-1)\pi x\right)}$$


y
f(x)
1
-3
-2
-1
1
2
3
x
1


y
$f_N(x)$
N >> 1.
x


y
$f_F(x)$
x