mth 235 L37

Plan: * Origins of Fourier Series.

* Periodic functions

* Orthogonality of Sines and cosines.

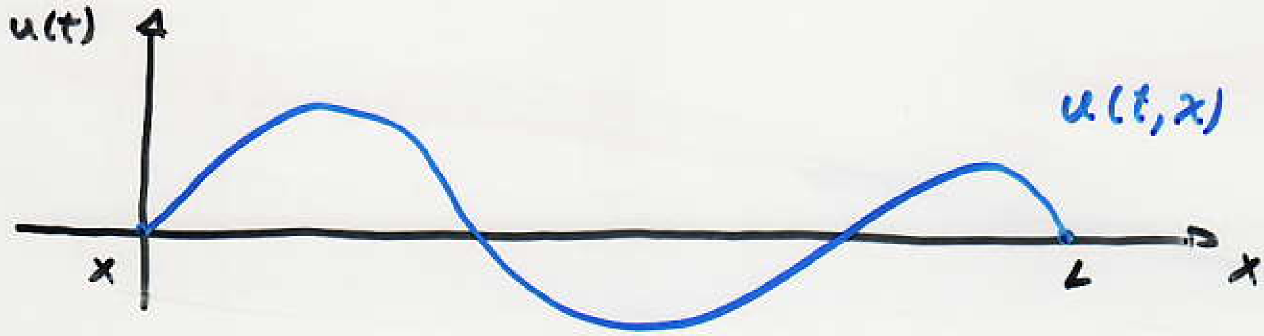
* Main result on Fourier series.

(10.2)

* Origins of Fourier Series

- Daniel Bernoulli (~ 1750)

He found solutions of the equation that describes waves propagating on a vibrating string.



Function u is solution of the wave eq.

$$\frac{\partial^2}{\partial t^2} u(t,x) = v^2 \frac{\partial^2}{\partial x^2} u(t,x)$$

I.C. $\quad u(0,x) = f(x) \qquad \frac{\partial}{\partial t} u(0,x) = 0$

B.C. $\quad u(t,0) = 0 \qquad u(t,L) = 0$

v: constant (units of velocity)

$x \in [0, L]$, $t \in [0, \infty)$

Bernoulli found particular solutions to this wave eq.

$$\boxed{u_n(t,x) = \sin\left(\frac{n\pi}{L}x\right)\cos\left(v\,\frac{n\pi}{L}t\right)}$$

Mathematical description of music.

with initial condition

$$\boxed{f_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$$

He also realized that

$$\boxed{U_N(t,x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(v\,\frac{n\pi}{L}t\right)}$$

is again a solution with initial cond.

$$\boxed{F_N(x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi}{L}x\right).}$$

$a_n \in \mathbb{R}$ arbitrary.

Bernoulli claimed that he had obtained all solutions to the wave eq.

He could not prove it; He did not say, given F_N, how to compute the coefficients a_n.

* Joseph Fourier (~ 1800) provided such formula for the a_n, while studying a different problem:

The heat transport in a solid material.

$$\frac{\partial}{\partial t} u(t,x) = k \frac{\partial^2}{\partial x^2} u(t,x), \quad k > 0$$

$$\text{I.C.} \quad u(0,x) = f(x) \qquad (2)$$

$$\text{B.C.} \quad u(t,0) = 0, \quad u(t,L) = 0$$

$$x \in [0,L], \quad t \in [0,\infty)$$

Fourier found the particular solution

$$\boxed{u_n(t,x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)}$$

with I.C.

$$\boxed{f_n(x) = \sin\left(\frac{n\pi}{L}x\right).}$$

He also realized that

$$\boxed{v_N(t,x) = \sum_{n=1}^{N} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)} \quad (3)$$

is again a solution with initial cond.

$$\boxed{F_N(x) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi}{L}x\right).}$$

* However, Fourier found a formula for a_n in terms of F_N.

$$\boxed{a_n = \frac{2}{L} \int_0^L F_N(x) \sin\left(\frac{n\pi}{L}x\right) dx}$$

[This formula is the key to prove that all sols. of (2) are given by (3).]

* Summary of the main result on Fourier Series

Every continuous τ-periodic function f can be expressed as an infinite linear combination of sine and cosine functions.

Given a function f, there exist constants a_n, b_n for $n = 0, 1, 2, \ldots$ such that

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2n\pi}{\tau} x\right) + b_n \sin\left(\frac{2\pi}{\tau} nx\right) \right]$$

satisfies

$$f_N(x) \xrightarrow[N \to \infty]{} f(x) \quad \text{for every } x \in \mathbb{R}.$$

Notation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi}{\tau} x\right) + b_n \sin\left(\frac{2n\pi}{\tau} x\right) \right] \qquad (4)$$

Main problem in our class:

$$\boxed{\text{Given a continuous, } \tau\text{-periodic function } f \text{, find formulas for } a_n, b_n \text{ such that Eq. (4) holds}}$$

* Periodic functions

Def: [A function $f : \mathbb{R} \to \mathbb{R}$ is called **periodic** iff exist $\tau > 0$ such that for all $x \in \mathbb{R}$ holds

$$\boxed{f(x+\tau) = f(x).} \qquad (5)$$]

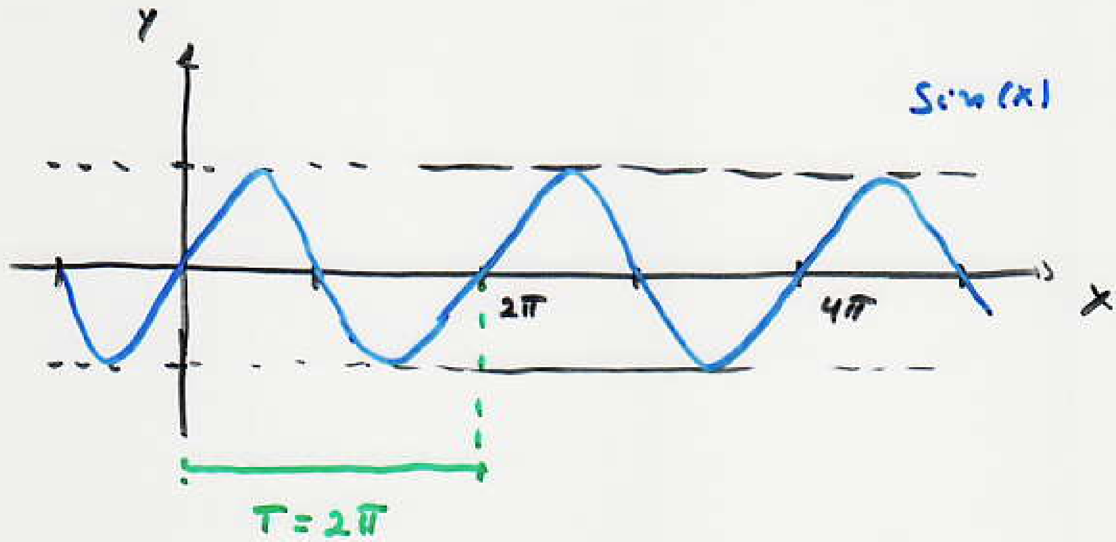
Remark: f is invariant under translations by τ.

Def: [The **period** T of a periodic function f is the smallest value of τ such that Eq. (5) holds.]

Notation: [A periodic function is also called T-periodic.]

* Examples

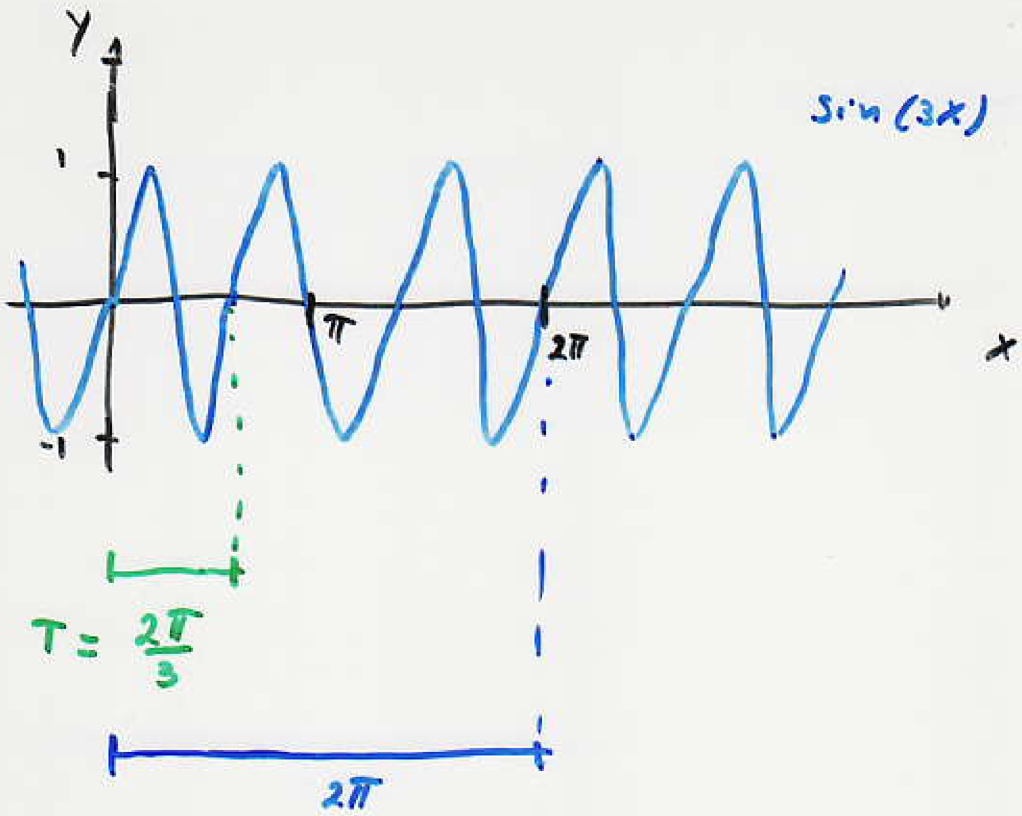
(1) $\boxed{f(x) = \sin(x)}$ is periodic with period $T = 2\pi$.



(2) $\boxed{f(x) = \cos(x)}$ is periodic with period $T = 2\pi$.

(3) $\boxed{f(x) = \sin(3x)}$ is periodic with period $T = \frac{2\pi}{3}$.

$$
\begin{aligned}
f\left(x + \tfrac{2}{3}\pi\right) &= \sin\left(3\left[x + \tfrac{2\pi}{3}\right]\right) \\
&= \sin(3x + 2\pi) \\
&= \sin(3x) \\
&= f(x).
\end{aligned}
$$



(4) $f(x) = \cos(\alpha x)$, $g(x) = \sin(\alpha x)$

are periodic with

period $T = \frac{2\pi}{\alpha}$.

Propos. [A linear combination of T-periodic functions is again T-periodic.]

Proof: If $f(x+T) = f(x)$, $g(x+T) = g(x)$

then, $a f(x+T) + b g(x+T) = a f(x) + b g(x)$.

so $(af + bg)$ is T-periodic. ∎

Example: [$f(x) = 2\sin(3x) + 7\cos(3x)$

is periodic with period $T = \frac{2\pi}{3}$]

Propos. [$f(x) = \cos\left(\frac{2\pi}{T} n x\right)$

$g(x) = \sin\left(\frac{2\pi}{T} n x\right)$, n integer

are periodic functions

with period T.]

Proof:

$$\boxed{f(x+T)} = \cos\left(\frac{2\pi n}{T}(x+T)\right)$$

$$= \cos\left(\frac{2\pi n x}{T} + 2\pi n\right)$$

$$= \cos\left(\frac{2\pi n x}{T}\right)$$

$$= \boxed{f(x)}.$$

∎

Corollary: Any function f given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)\right]} \quad (6)$$

is periodic with period T

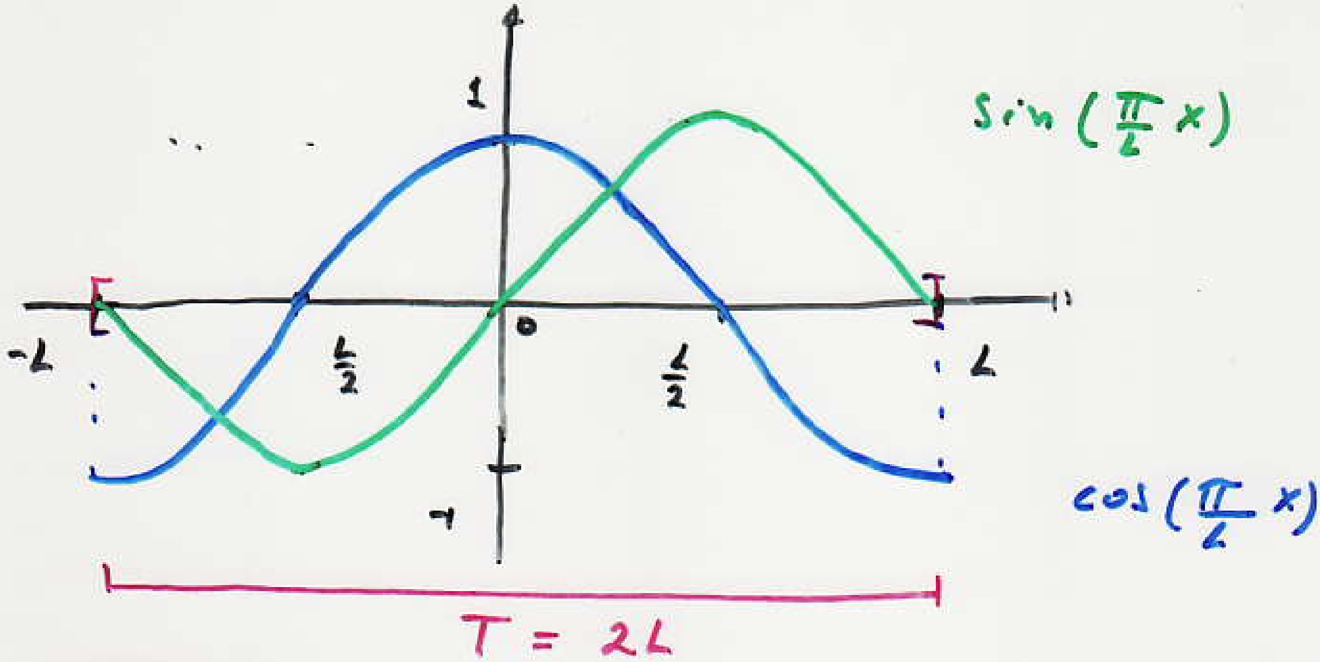
Remark: [We now show that the converse statement is True:]

> Every T-periodic continuous function f can be expressed as in Eq. (6) for appropriate values of a_n, b_n.

* Orthogonality relations for sine and cosine

We work in the following domain:

$D = [-L, L]$.



Thrm:

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0.$$

Proof: use the formulas,

$$\boxed{\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]}$$

$$\boxed{\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]}$$

$$\boxed{\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]}$$

$$\boxed{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx} =$$

$$= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n+m)\pi x}{L}\right) dx + \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n-m)\pi x}{L}\right)dx \qquad (7)$$

If $\qquad m \neq \pm n$

$$= \frac{1}{2}\frac{L}{(n+m)\pi} \underbrace{\sin\left(\frac{(n+m)\pi x}{L}\right)\Bigg|_{-L}^{L}}_{=0}$$

$$+ \frac{1}{2}\frac{L}{(n-m)\pi} \underbrace{\sin\left(\frac{(n-m)\pi x}{L}\right)\Bigg|_{-L}^{L}}_{=0}$$

$$\boxed{= 0.} \quad m \neq \pm n$$

If $n = m$, (7) $\Rightarrow$

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{2n\pi}{L} x\right) dx$$

$$+ \frac{1}{2}\int_{-L}^{L} dx$$

$$= \frac{1}{2}\left(\frac{L}{2n\pi}\right) \underbrace{\sin\left(\frac{2n\pi}{L} x\right)\Big|_{-L}^{L}}_{=0} + \frac{1}{2}\left(L-(-L)\right)$$

$$= L \, .$$

<u>Exercise</u>: complete the proof.

* Main Result: The Fourier Series expansion Thrm.

Thrm: If function $f : [-L, L] \to \mathbb{R}$ is continuous;
Then f can be expressed as

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]} \quad (8)$$

With a_n, b_n, $n = 0, 1, \ldots$, given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad (9)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (10)$$

Furthermore, $f : [-L, L] \to \mathbb{R}$ can be extended as a $2L$-periodic function $f : \mathbb{R} \to \mathbb{R}$.