mth 235 L 36

Plan: * Review: Exam 3

* Sects: 6.1 – 6.6, 7.1 – 7.6, 7.8

* 5 problems, 50 min.

* Laplace Transforms table on page 317 included in the exam.

* Exam November 12, 2008.

(Problem 4) Find the general sol. of

$$\underline{x}' = A\,\underline{x}, \qquad A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}$$

Sketch a phase diagram (portrait).

Sol:

Eigenvalues:

$$P(\lambda) = \begin{vmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda+3) - 2$$

$$P(\lambda) = \lambda^2 + 3\lambda + 2\lambda + 6 - 2$$

$$P(\lambda) = \lambda^2 + 5\lambda + 4 = 0$$

$$\lambda_\pm = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm \sqrt{9}}{2}$$

$$\lambda_\pm = \frac{-5 \pm 3}{2} \Rightarrow \boxed{\lambda_+ = -1} \quad \boxed{\lambda_- = -4}$$

so: $$\boxed{\lambda_- < \lambda_+ < 0}$$

Eigenvectors:

$$\boxed{\lambda_+ = -1}$$

$$A + I = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \begin{matrix} \leftarrow -1 \\ \leftarrow \sqrt{2} \end{matrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad 2 V_1 = \sqrt{2}\, V_2.$

Choosing: $\quad V_2 = 2 \,, \quad V_1 = \sqrt{2}$

$$\Rightarrow \boxed{\underline{V}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \ \lambda_+ = -1}$$

$$\boxed{\lambda_- = -4}$$

$$A + 4I = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \begin{matrix} \\ \leftarrow \frac{1}{\sqrt{2}} \end{matrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad V_1 = -\sqrt{2}\, V_2$

Choosing: $V_2 = 1, \ V_1 = -\sqrt{2}$

$$\Rightarrow \boxed{\underline{V}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}, \ \lambda_- = -4}

Fundamental solutions:

$$\underline{x}^{(+)} = \underline{v}^{(+)} e^{\lambda_+ t} \Rightarrow \boxed{\underline{x}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t}}$$

$$\underline{x}^{(-)} = \underline{v}^{(-)} e^{\lambda_- t} \Rightarrow \boxed{\underline{x}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}}$$

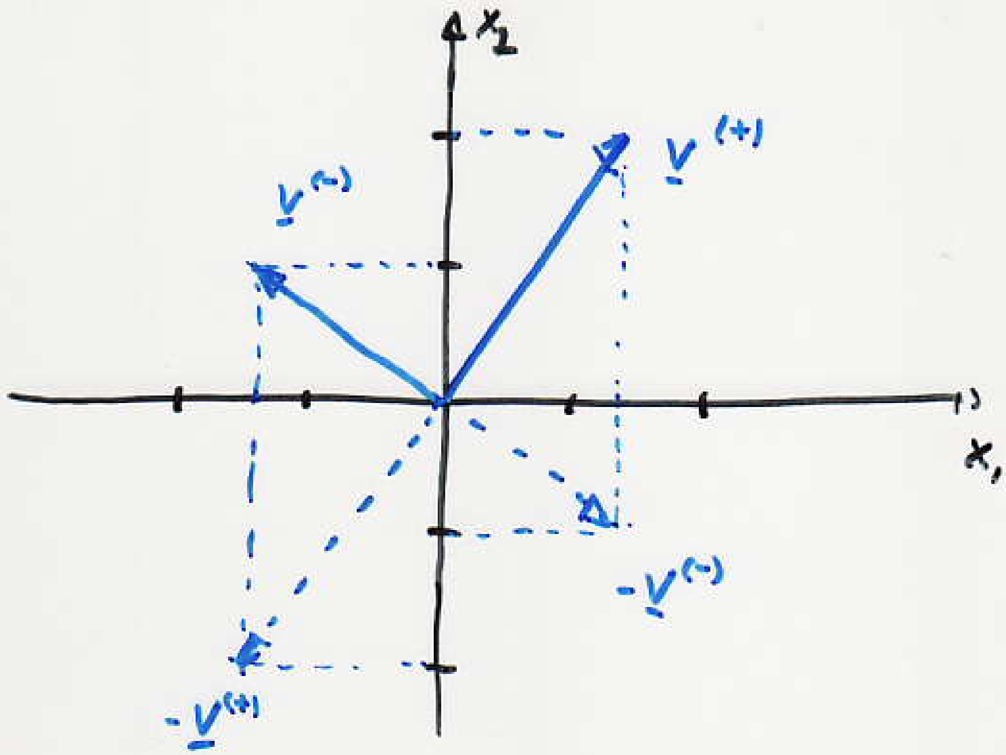
The general sol is

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}}$$

The phase diagram : case : $\lambda_- < \lambda_+ < 0$.

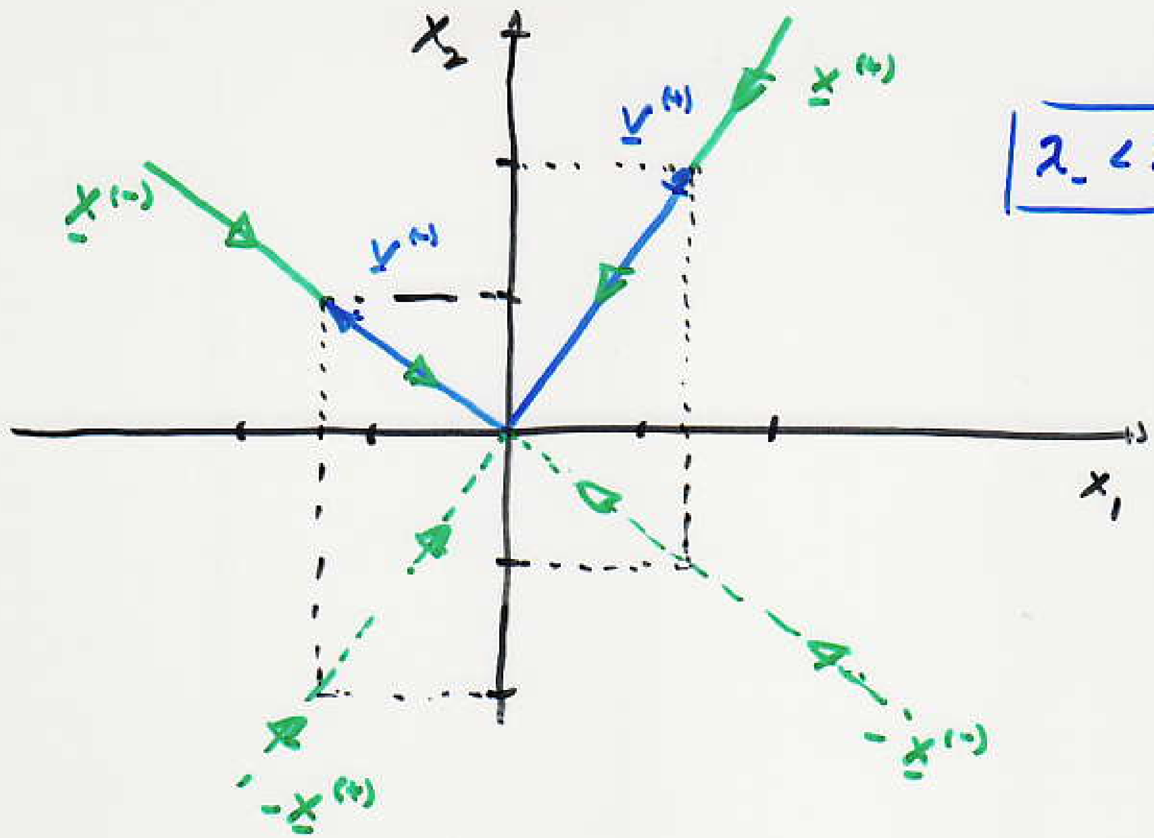
(1) plot $\pm \underline{v}^{(+)}$ and $\pm \underline{v}^{(-)}$

$$\underline{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \quad \underline{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

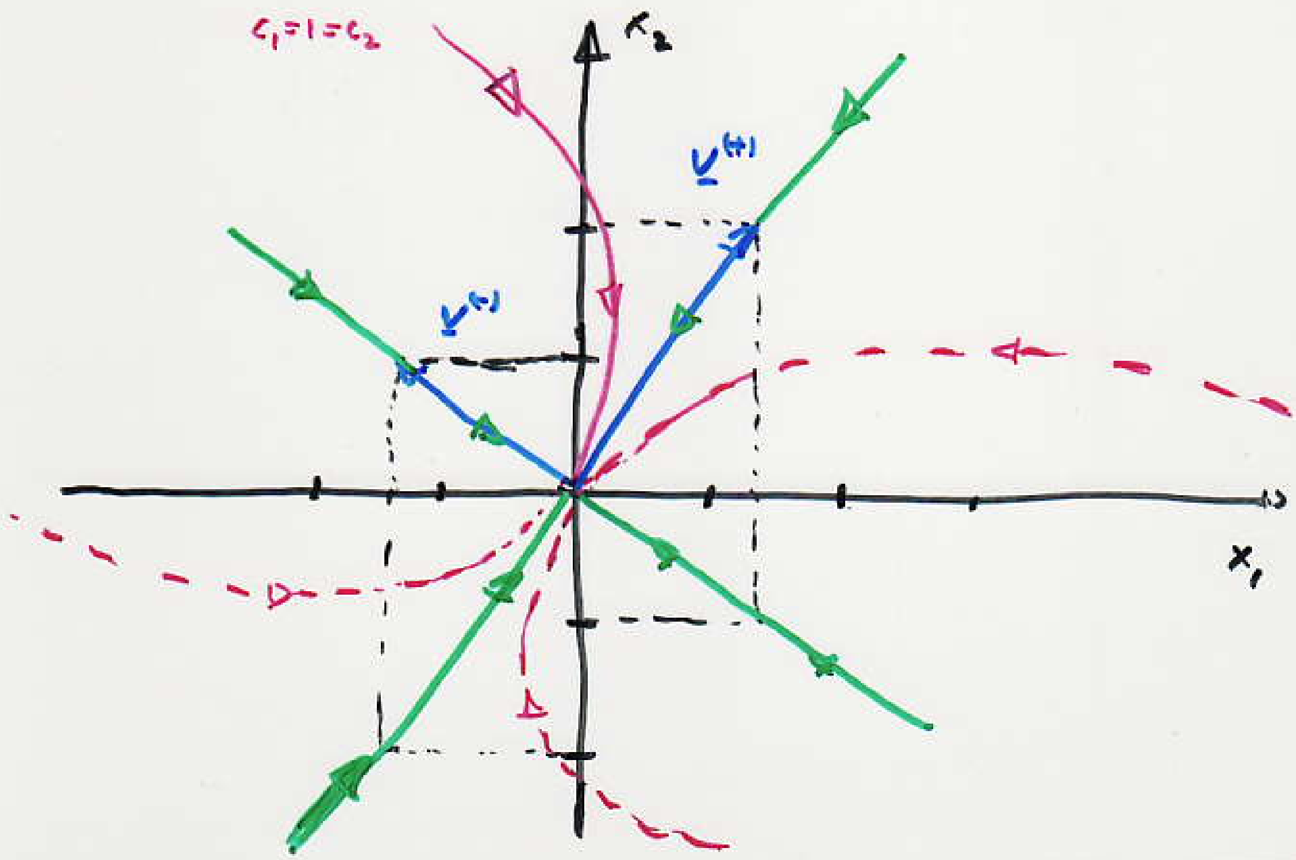


(2) Plot sols. proportional to $\underline{x}^{(\pm)} = \underline{v}^{(\pm)} e^{\lambda_\pm t}$.

$\boxed{\lambda_- < \lambda_+ < 0}$



(3) Plot few more sols: $\underline{x}(t) = c_1 \underline{v}^{(+)} e^{-t} + c_2 \underline{v}^{(-)} e^{-4t}$



* *Example*

consider the case:

$$\underline{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \quad \underline{v}^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

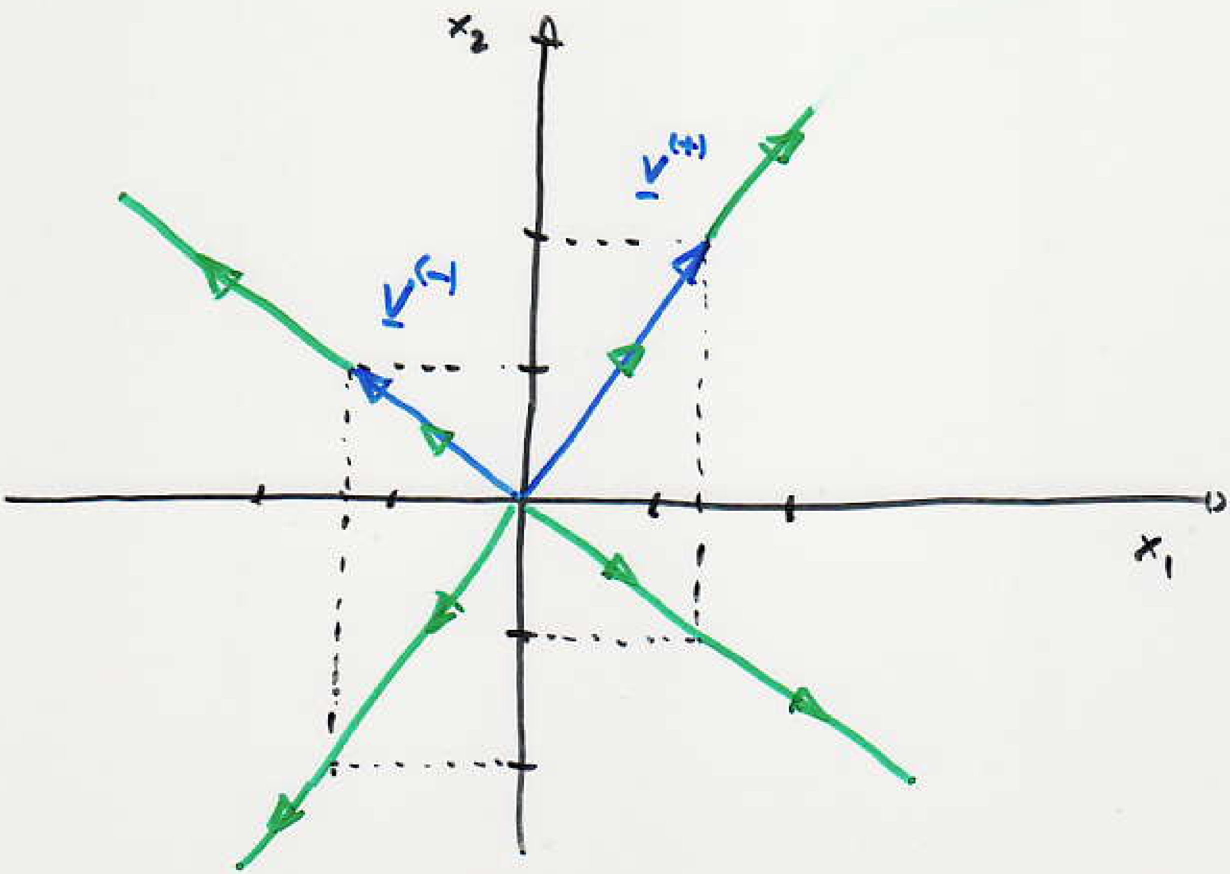
and $\lambda_+ = 4, \quad \lambda_- = 1$

so: $\boxed{\lambda_+ > \lambda_- > 0.}$

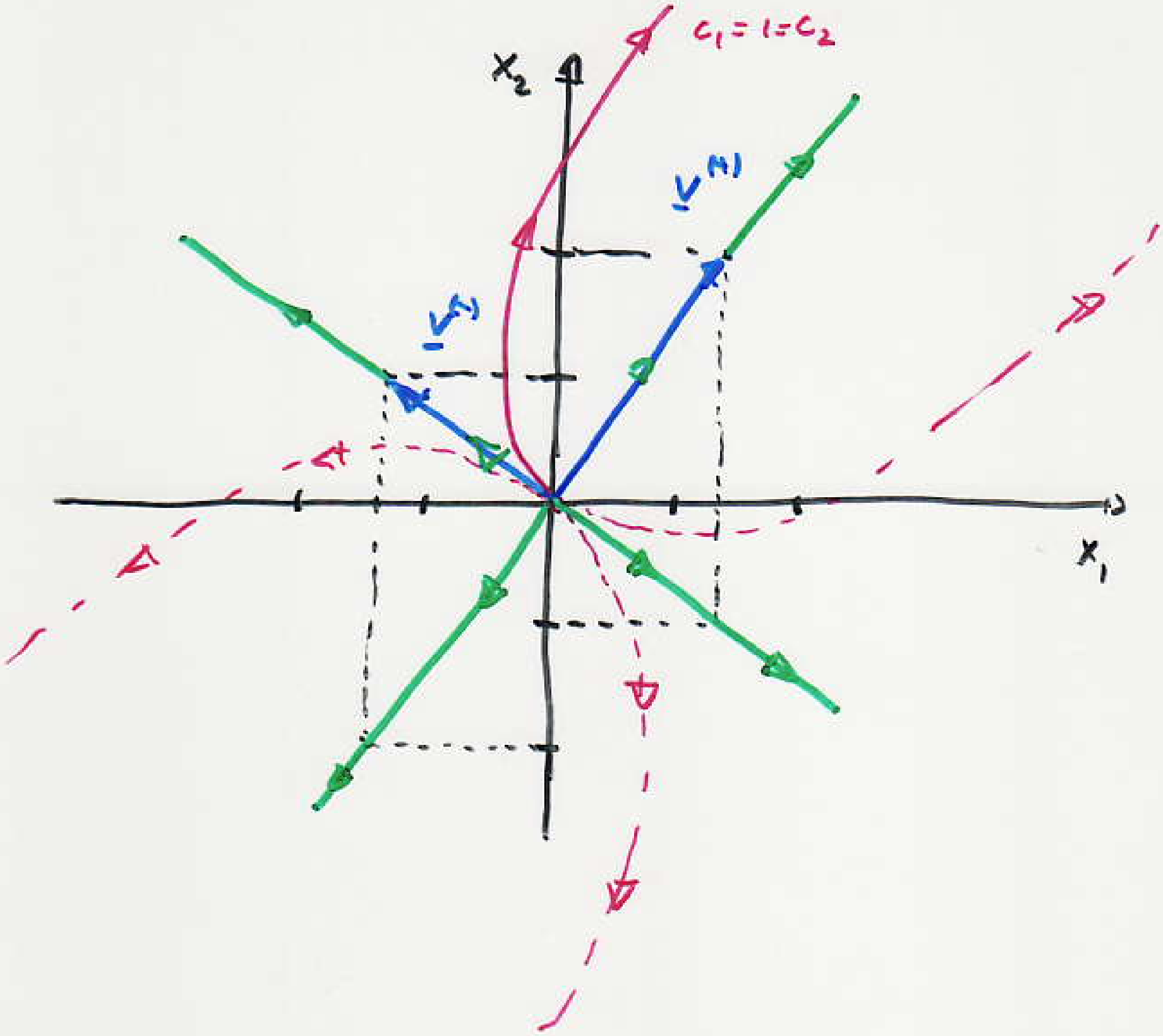
Sketch a phase diagram.

Sol:

(1), (2) Plot $v^{(+)}$, $v^{(-)}$ and sols. proportional to $\underline{x}^{(\pm)} = \underline{v}^{(\pm)} e^{\lambda_\pm t}$



(3) plot few more sols $\underline{x} = c_1 \underline{v}^{(4)} e^{4t} + c_2 \underline{v}^{(1)} e^{t}$



* *Example* : Consider the case

$$\underline{v}^{(+)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}, \quad v^{(-)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

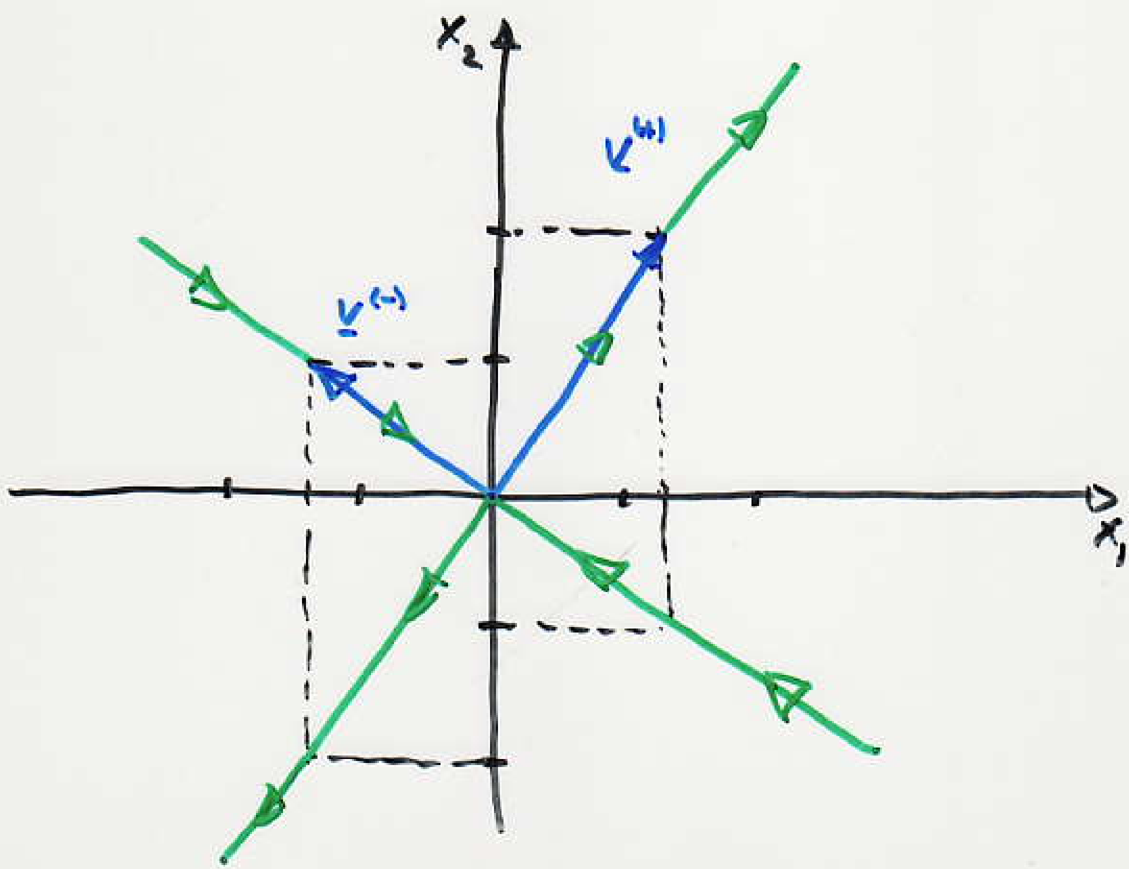
and $\lambda_+ = 4, \quad \lambda_- = -1$

so : $\boxed{\lambda_+ > 0 > \lambda_-}$

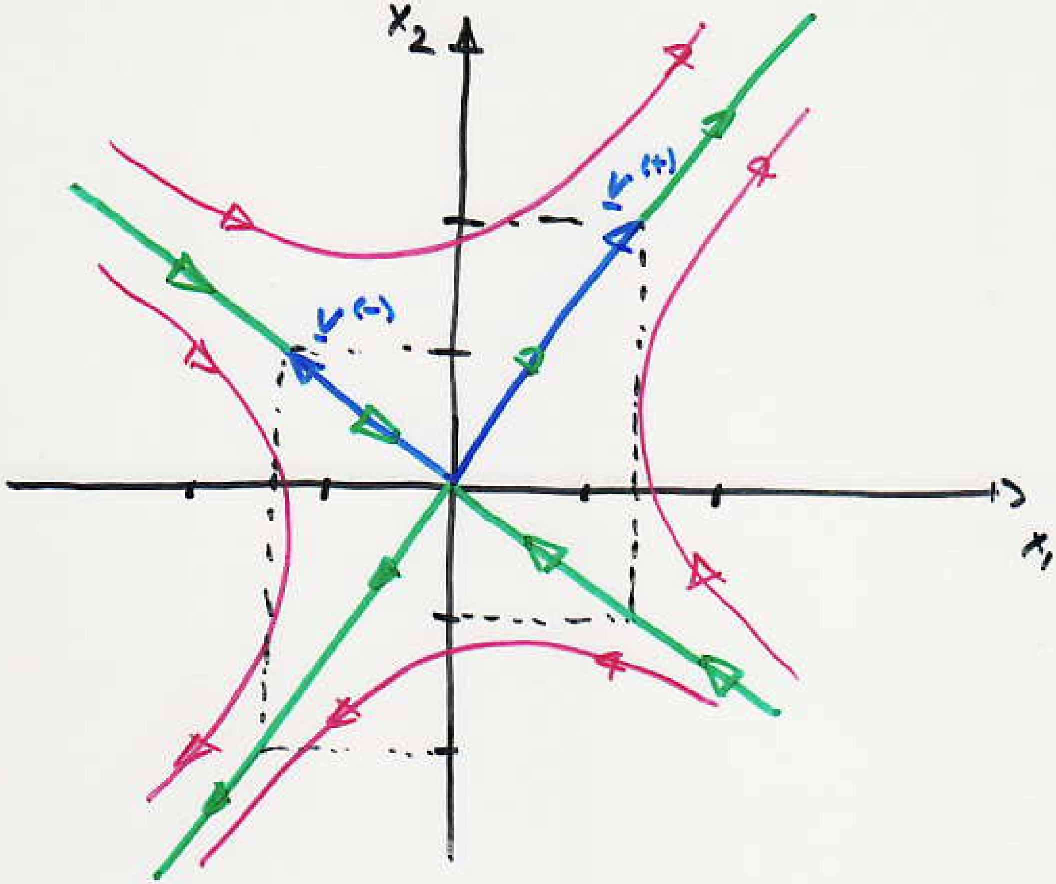
Sketch a phase diagram.

Sol:

(1),(2) Plot $\underline{v}^{(+)}$, $v^{(-)}$ and sols. proportional to: $\underline{x}^{(\pm)} = \underline{v}^{(\pm)} e^{\lambda_\pm t}$.



(3) plot few more sols.



Example : Find $\underline{x}$ sol. of IVP

$$\underline{x}' = A\,\underline{x}\,, \quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix},$$

$$\underline{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Sketch a phase diagram containing sols. proportional to fund. sols.

Sol :

Eigenvalues

$$p(\lambda) = \begin{vmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+3) + 4$$

$$p(\lambda) = \lambda^2 + 3\lambda - \lambda - 3 + 4 = \lambda^2 + 2\lambda + 1$$

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_\pm = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

$$\boxed{\lambda_\pm = -1}$$ repeated eigenvalue.

Eigenvectors

$\lambda = -1$

$$A + I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\left.\begin{array}{l} v_1 = 2 v_2 \\ \text{choosing: } v_2 = 1, \; v_1 = 2 \end{array}\right\} \Rightarrow \boxed{\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \lambda = -1}$$

Find $\underline{w}$ sol. of $(A + I)\,\underline{w} = \underline{v}$.

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 4 & 2 \\ -1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 1 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad w_1 = 2 w_2 - 1 \quad \Rightarrow \quad \underline{w} = \begin{bmatrix} 2 w_2 - 1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\text{choosing } w_2 = 0 \quad \Rightarrow \quad \boxed{\underline{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

- Fundamental sols:

$$\boxed{\begin{aligned} \underset{\sim}{x}^{(1)} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \\ \underset{\sim}{x}^{(2)} &= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t} \end{aligned}}$$

- General solution

$$\boxed{\underset{\sim}{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}}$$

I. C.

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underset{\sim}{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\frac{1}{0+1} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^{-1}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

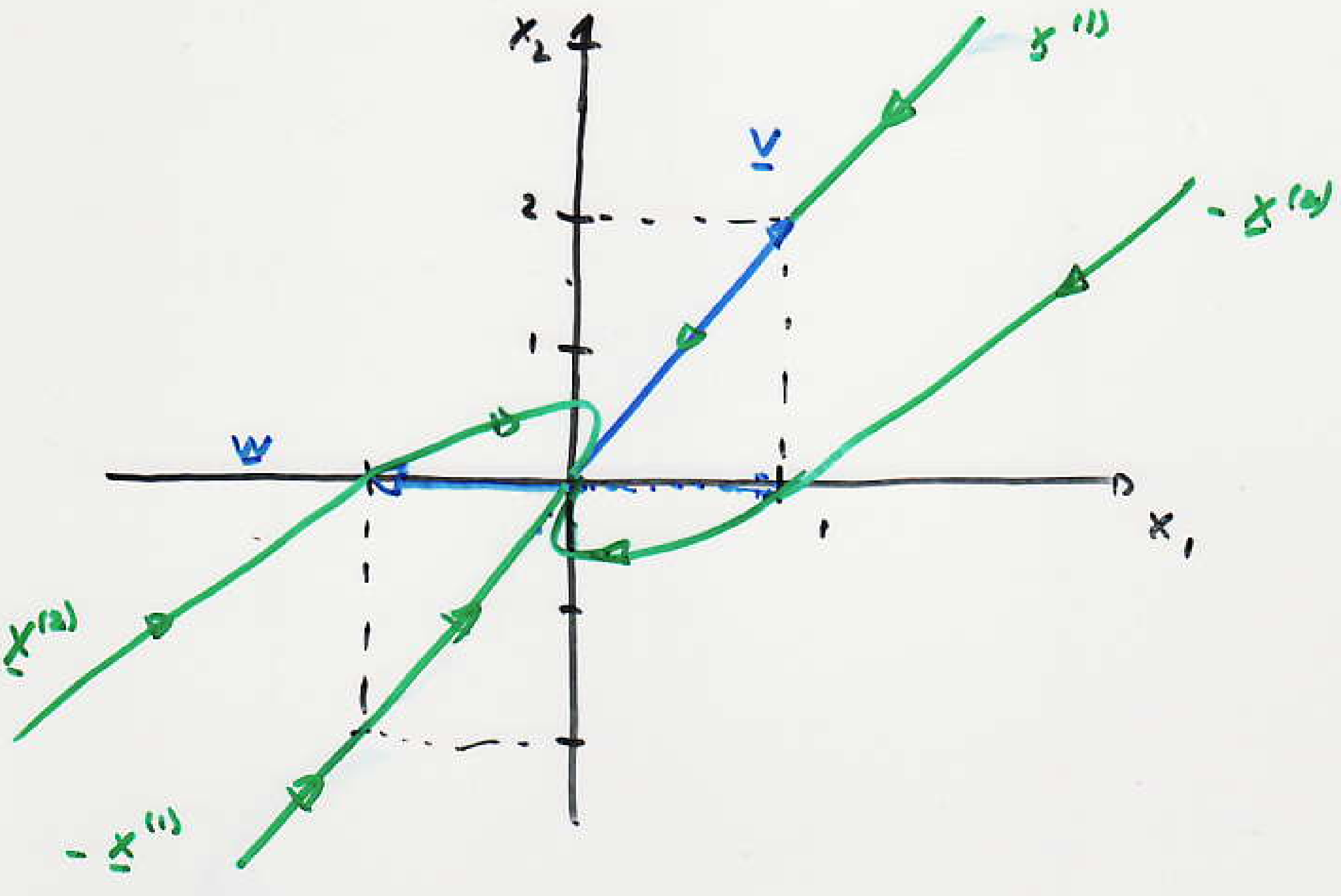
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} c_1 = 3 \\ c_2 = 5 \end{matrix}}$$

$$\boxed{\underset{\sim}{x}(t) = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 5 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{-t}}$$

* Phase diagram for $\pm \underset{\sim}{x}^{(1)}$ and $\pm \underset{\sim}{x}^{(2)}$

(1) Plot $\pm \underset{\sim}{x}^{(1)} = \pm \underset{\sim}{v} \, e^{\lambda t}$ $\boxed{\lambda < 0}$

$\pm \underset{\sim}{x}^{(2)} = \pm (\underset{\sim}{v} t + \underset{\sim}{w}) \, e^{\lambda t}$



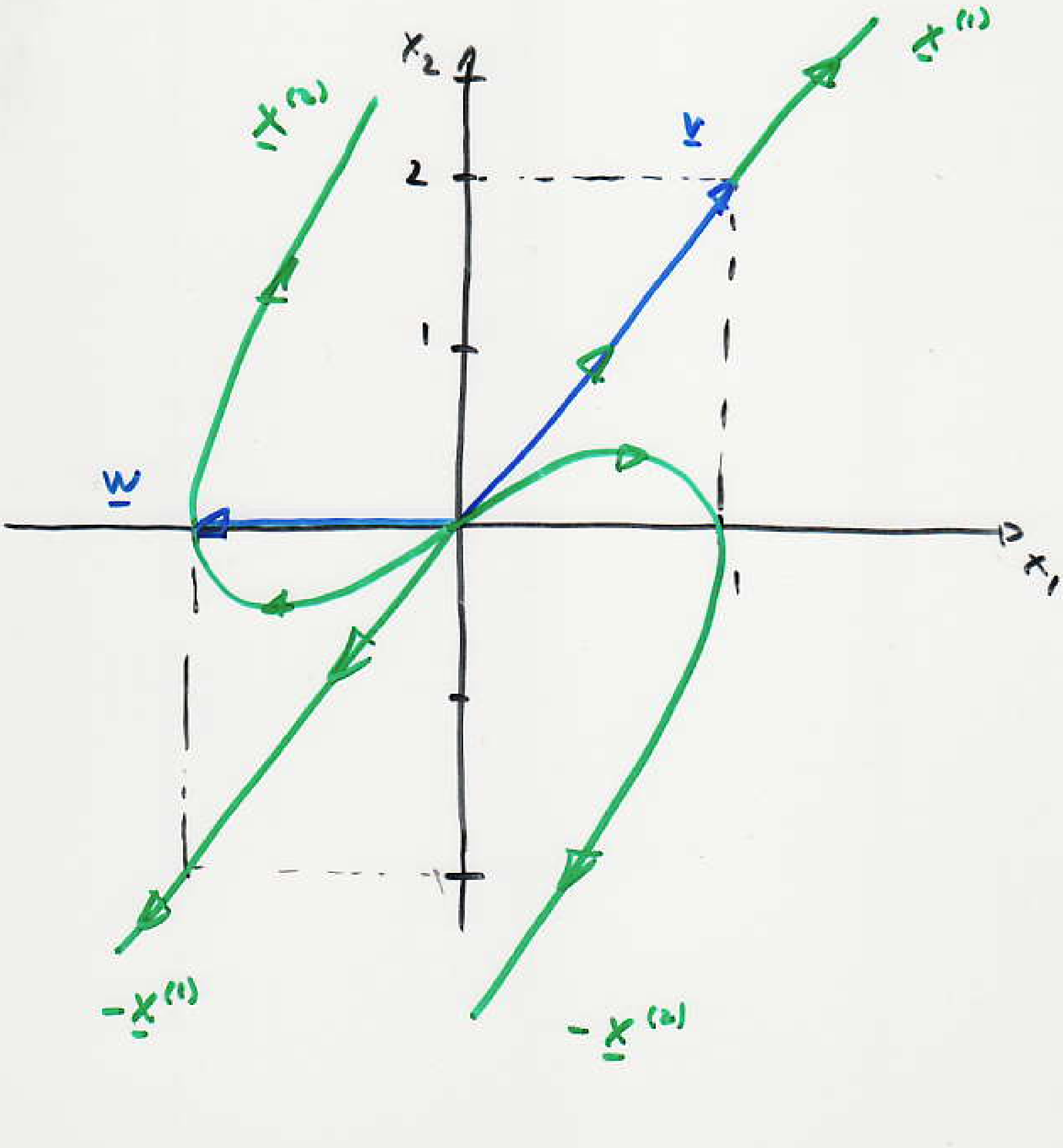
* Example

case: $\boxed{\lambda > 0}$ with $\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Plot $\pm \underline{x}^{(1)} = \pm \underline{v}\, e^{-\lambda t}$

$\pm \underline{x}^{(2)} = \pm (\underline{v}\, t + \underline{w})\, e^{-\lambda t}$

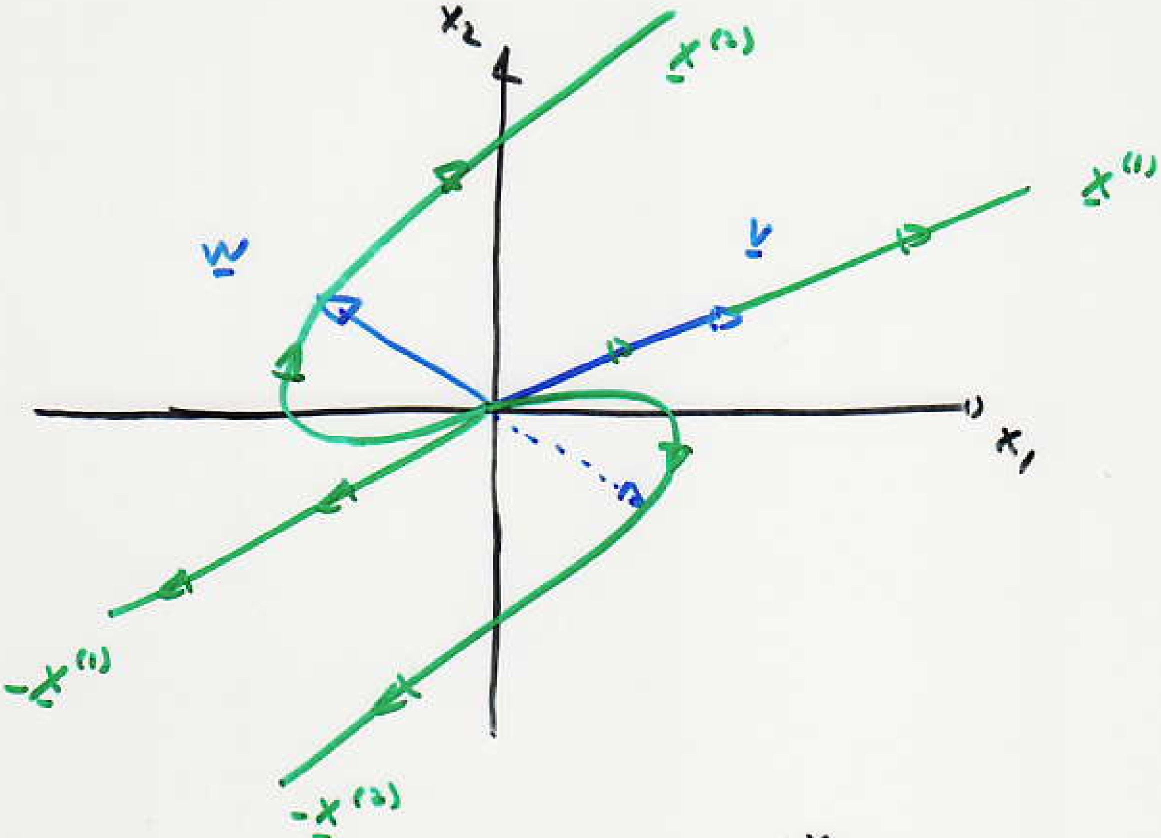
Sol.



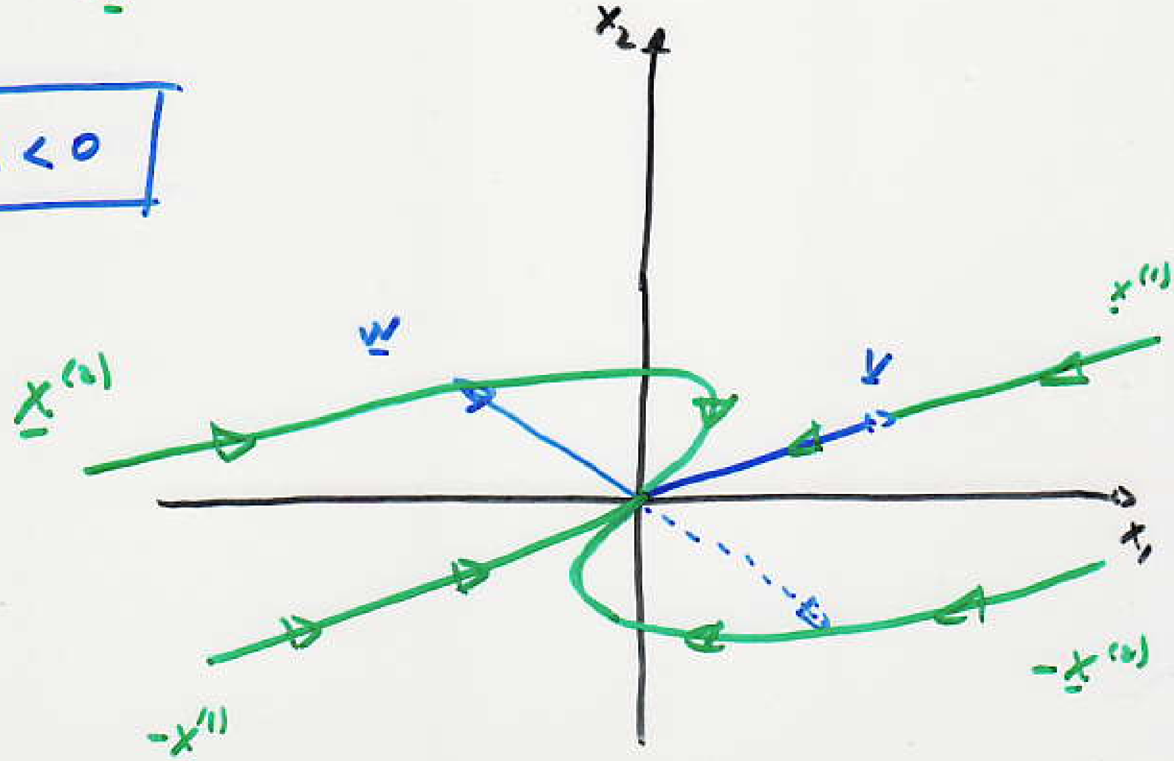
Summary : $\boxed{\lambda > 0}$

$$\underline{x}^{(1)} = \underline{v}\, e^{\lambda t}$$

$$x^{(2)} = (\underline{v}\, t + \underline{w})\, e^{\lambda t}$$



$\boxed{\lambda < 0}$



* Example : Complex eigenvalues

Plot the phase diagram of a system $\underline{x}' = A\,\underline{x}$ where matrix A has eigenvalues $\lambda_\pm = \alpha \pm \beta i$, $\beta \neq 0$ and eigenvectors: $\underline{v}^{(\pm)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \pm \begin{bmatrix} 1 \\ 3 \end{bmatrix} i$

Sol.

$$\underline{v}^{\pm} = \underline{a} \pm \underline{b}\,i \Rightarrow \underline{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

So, real valued fund. sols. are obtained from: real and imaginary parts of:

$$\underline{x}^{(+)} = (\underline{a} + \underline{b}\,i)\, e^{\lambda_+ t}$$

$$= (\underline{a} + \underline{b}\,i)\, e^{\alpha t} e^{i\beta t}$$

$$= (\underline{a} + \underline{b}\,i)\, \left(\cos(\beta t) + i \sin(\beta t)\right) e^{\alpha t}$$

$$\boxed{\begin{aligned} \underline{x}^{(+)} &= \left[\underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)\right] e^{\alpha t} \\ &+ i\left[\underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)\right] e^{\alpha t}. \end{aligned}}$$

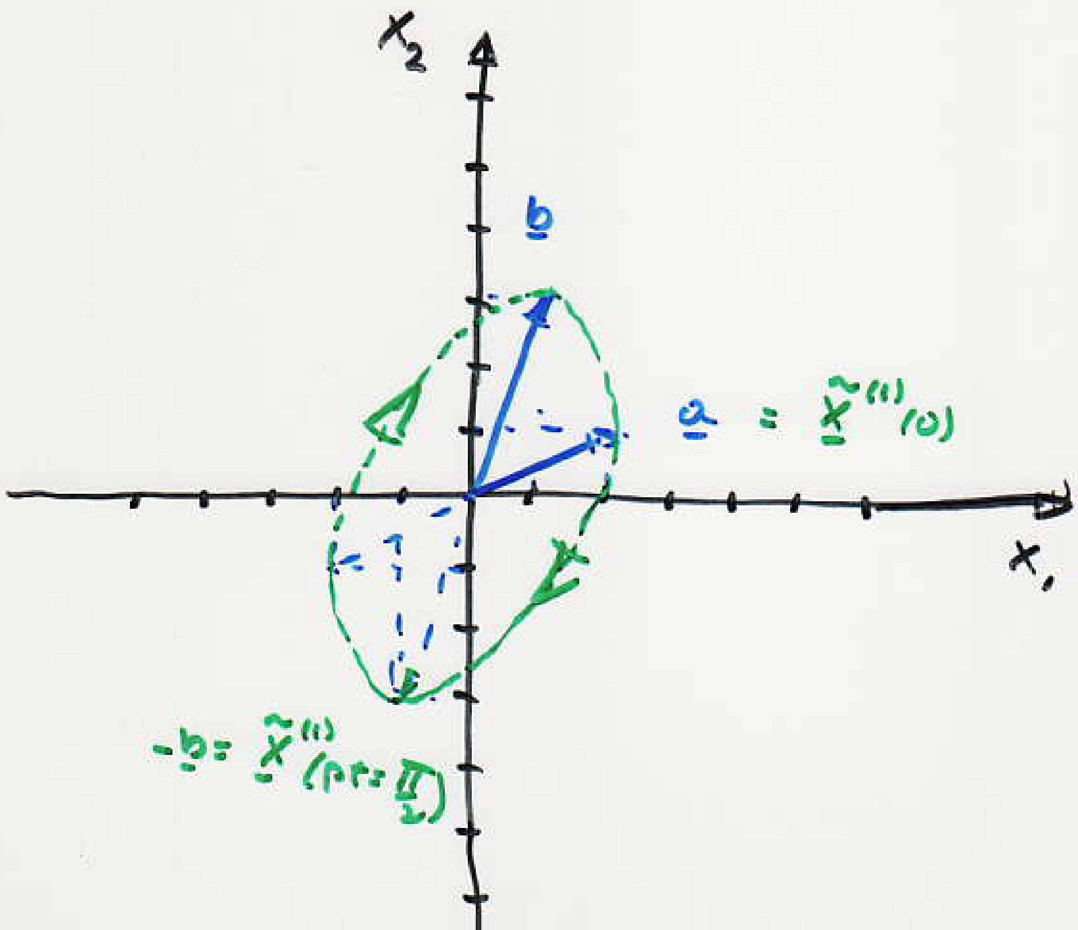
$$\boxed{\begin{aligned} \underline{x}^{(1)} &= \left[\underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)\right] e^{\alpha t} \\ \underline{x}^{(2)} &= \left[\underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)\right] e^{\alpha t} \end{aligned}}$$

$$\underline{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(1) plot

$$\tilde{\underline{x}}^{(1)} = \underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)$$

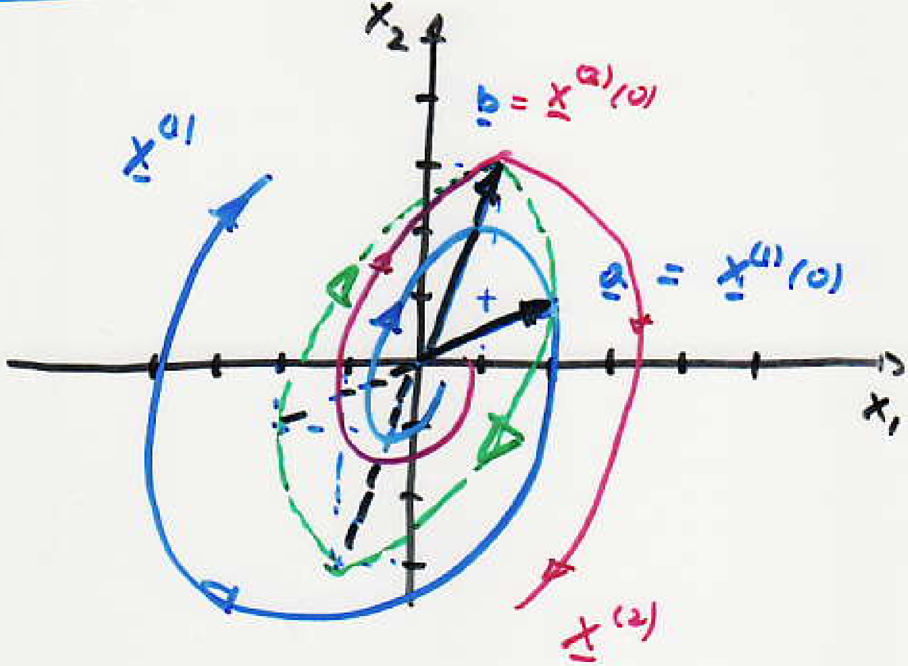
$$\tilde{\underline{x}}^{(2)} = \underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)$$



(2) Plot $\underline{x}^{(1)} = \underline{\tilde{x}}^{(1)} e^{\alpha t}$

$\underline{x}^{(2)} = \underline{\tilde{x}}^{(2)} e^{\alpha t}$.

$\boxed{\text{Case } \alpha > 0}$



$\boxed{\text{Case } \alpha < 0}$