mth 235 L33

Plan: * Review: Exam 3

* Sects. 6.1 - 6.6, 7.1 - 7.6, 7.8

* 5 ~ 6 problems, 50 min.

* Laplace Transforms table on page 317 included in exam.

* Exam : November 11, 2008.

(Prb. 2) $\left[\begin{array}{l} \text{Use L.T. to find } y \text{ sol. of} \\ y'' - 2y' + 2y = \delta(t-2) \\ y(0) = 1, \quad y'(0) = 3 \end{array}\right]$

Sol:

$$\mathcal{L}[y''] - 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\delta(t-2)]$$

$$\boxed{\mathcal{L}[\delta(t-2)] = e^{-2s}}$$

$$\boxed{\begin{array}{l} \mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \\ \mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0) \end{array}}$$

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s\,y(0) - y'(0) - 2(-y(0)) = e^{-2s}$$

(I.C.)

$$(s^2 - 2s + 2)\,\mathcal{L}[y] - s - 3 + 2 = e^{-2s}$$

$$(s^2 - 2s + 2)\, \mathcal{L}[y] = s + 1 + e^{-2s}$$

$$\boxed{\mathcal{L}[y] = \frac{(s+1)}{(s^2-2s+2)} + e^{-2s}\,\frac{1}{(s^2-2s+2)}}$$

$$s^2 - 2s + 2 = 0 \Rightarrow s_{\pm} = \frac{2 \pm \sqrt{4-8}}{2}$$ complex roots.

complete the square.

$$s^2 - 2s + 2 = (s^2 - 2s + 1) - 1 + 2$$

$$= (s-1)^2 + 1$$

$$\mathcal{L}[y] = \frac{(s+1)}{(s-1)^2+1} + e^{-2s}\,\frac{1}{(s-1)^2+1}$$

$$= \frac{(s-1)+1+1}{(s-1)^2+1} + e^{-2s}\,\frac{1}{(s-1)^2+1}$$

$$\boxed{\mathcal{L}[y] = \frac{(s-1)}{(s-1)^2+1} + 2\frac{1}{(s-1)^2+1} + e^{-2s}\frac{1}{(s-1)^2+1}}$$

$$F_1(s) = \frac{s}{s^2+1} = \mathcal{L}[\cos(t)]$$

$$F_2(s) = \frac{1}{s^2+1} = \mathcal{L}[\sin(t)]$$

Table.

$$\frac{(s-1)}{(s-1)^2+1} = F_1(s-1) = \mathcal{L}[e^t\cos(t)]$$

$$\frac{1}{(s-1)^2+1} = F_2(s-1) = \mathcal{L}[e^t\sin(t)]$$

$$\boxed{\begin{aligned}\mathcal{L}[y] &= \mathcal{L}[e^t\cos(t)] + 2\,\mathcal{L}[e^t\sin(t)] \\ &\quad + e^{-2s}\,\mathcal{L}[e^t\sin(t)]\end{aligned}}$$

$$\boxed{\begin{aligned}\mathcal{L}[y] &= \mathcal{L}[\, e^{t}\cos(t) + 2\,e^{t}\sin(t) \,] \\ &\quad + \mathcal{L}[\, u(t-2)\; e^{(t-2)}\; \sin(t-2) \,]\end{aligned}}$$

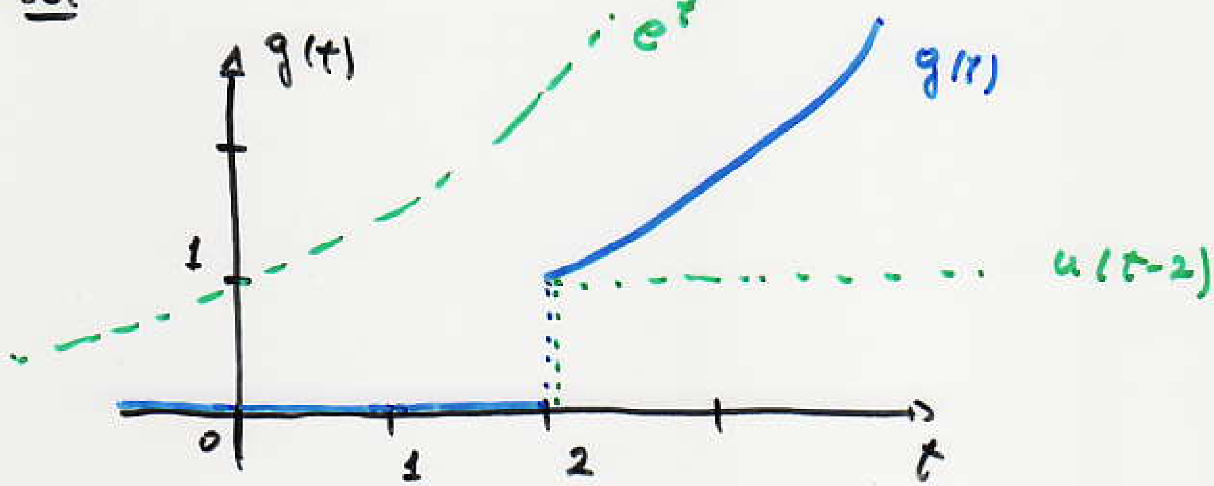
$$\boxed{\begin{aligned}y(t) &= e^{t}\,(\cos(t) + 2\sin(t)) \\ &\quad + u(t-2)\; e^{(t-2)}\; \sin(t-2)\end{aligned}}$$

(Problem 3) Sketch the graph of g and use L.T. to find y sol. of

$$y'' + 3y = g(t) \quad , \quad y(0) = y'(0) = 0$$

$$g(t) = \begin{cases} 0 & t < 2 \\ e^{(t-2)} & t \geq 2 \end{cases}$$

Sol:



- Express g in terms of step functions:

$$\boxed{g(t) = u(t-2)\, e^{(t-2)}}$$

$$\mathcal{L}[g] = e^{-2s}\, \mathcal{L}[e^t]$$

$$\boxed{\mathcal{L}[g] = \frac{e^{-2s}}{(s-1)}}$$

$$\boxed{\mathcal{L}[y''] + 3\,\mathcal{L}[y] = \mathcal{L}[g]}$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)$$

I.C. $\Rightarrow$ $\mathcal{L}[y''] = s^2\,\mathcal{L}[y]$.

$$(s^2+3)\,\mathcal{L}[y] = \frac{e^{-2s}}{(s-1)}$$

$$\boxed{\mathcal{L}[y] = e^{-2s}\,\frac{1}{(s-1)\,(s^2+3)}}$$

Partial fractions.

$$\frac{1}{(s-1)\,(s^2+3)} = \frac{a}{(s-1)} + \frac{bs+c}{(s^2+3)}$$

$$= \frac{a\,(s^2+3) + (bs+c)\,(s-1)}{(s-1)\,(s^2+3)}$$

$$1 = as^2 + a3 + bs^2 + cs - bs - c$$

$$\boxed{1 = (a+b)\,s^2 + (c-b)\,s + (3a-c)}$$

$$\left.\begin{array}{l} a+b=0 \\ c-b=0 \quad \Rightarrow \underline{b=c} \\ 3a-c=1 \end{array}\right| \Rightarrow \left.\begin{array}{l} a+c=0 \quad \Rightarrow \underline{c=-a} \\ 3a-c=1 \end{array}\right] \Rightarrow$$

$$3a + a = 1 \Rightarrow \boxed{a = \frac{1}{4}} \quad \boxed{b = -\frac{1}{4}} \quad \boxed{c = -\frac{1}{4}}$$

$$\boxed{\frac{1}{(s-1)\,(s^2+3)} = \frac{1}{4}\,\frac{1}{(s-1)} - \frac{1}{4}\,\frac{(s+1)}{s^2+3}}$$

$$\mathcal{L}[Y] = \frac{e^{-2s}}{4}\left[\frac{1}{(s-1)} - \frac{s}{(s^2+3)} - \frac{1}{(s^2+3)}\right]$$

$$\mathcal{L}[y] = \frac{e^{-2s}}{4}\Big[\mathcal{L}[e^t] - \mathcal{L}[\cos(\sqrt{3}\,t)]$$

$$- \mathcal{L}[\sin(\sqrt{3}\,t)]\Big]$$

$$\boxed{h(t) = \frac{1}{4}\left[e^t - \cos(\sqrt{3}\,t) - \sin(\sqrt{3}\,t)\right]}$$

$$\mathcal{L}[y] = e^{-2s}\,\mathcal{L}[h(t)]$$

$$= \mathcal{L}[u(t-2)\,h(t-2)]$$

$$\boxed{y(t) = u(t-2)\,h(t-2)}$$ ✓

or:

$$\boxed{y(t) = \frac{u(t-2)}{4}\Big[e^{(t-2)} - \cos(\sqrt{3}\,(t-2)) - \sin(\sqrt{3}\,(t-2))\Big]}$$

Extra Problem: use convolutions to find $f(t)$ satisfying:

$$\mathcal{L}[f(t)] = \frac{e^{-2s}}{(s-1)(s^2+3)}$$

Sol:

Possibility 1:

$$\mathcal{L}[f] = \left(\frac{e^{-2s}}{(s^2+3)}\right)\left(\frac{1}{(s-1)}\right)$$

$$\frac{e^{-2s}}{s^2+3} = e^{-2s}\,\mathcal{L}[\sin(\sqrt{3}\,t)]$$

$$= \mathcal{L}[u(t-2)\,\sin(\sqrt{3}\,(t-2))]$$

$$\frac{1}{(s-1)} = \mathcal{L}[e^t]$$

$$\boxed{\mathcal{L}[f] = \mathcal{L}[e^t] \; \mathcal{L}[u(t-2)\sin(\sqrt{3}(t-2))]}$$

$$\boxed{f(t) = \int_0^t e^{(t-\tau)} \; u(\tau-2)\sin(\sqrt{3}(\tau-2))\,d\tau}$$

Possibility 2:

$$\boxed{\mathcal{L}[f] = \left(\frac{1}{s^2+3}\right)\left(\frac{e^{-2s}}{(s-1)}\right)}$$

$$\boxed{\frac{1}{s^2+3} = \mathcal{L}[\sin(\sqrt{3}t)]}$$

$$\frac{e^{-2s}}{(s-1)} = e^{-2s} \; \mathcal{L}[e^t]$$

$$= \mathcal{L}[u(t-2)\,e^{(t-2)}]$$

$$\boxed{\mathcal{L}[f] = \mathcal{L}[\sin(\sqrt{3}\,t)]\ \mathcal{L}[u(t-2)\,e^{(t-2)}]}$$

$$\boxed{f(t) = \int_0^t \sin(\sqrt{3}\,(t-\tau))\ u(\tau-2)\,e^{(\tau-2)}\,d\tau}$$

Notice: We also found:

$$\boxed{f(t) = u(t-2)\ h(t-2)}$$

$$\boxed{h(t) = \frac{1}{4}\left[e^{t} - \cos(\sqrt{3}\,t) - \sin(\sqrt{3}\,t)\right]}$$

(Prblm 4) We change Prb. 4 into:

Find the real-valued gen. sol. of

$$\underline{x}' = A\,\underline{x} \quad , \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Sol:

- Eigenvalues of A.

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 + 4 = 0$$

$$(\lambda-1)^2 = -4 \Rightarrow \lambda_\pm - 1 = \pm 2i \Rightarrow$$

$$\boxed{\lambda_\pm = 1 \pm 2i}$$

- Eigenvectors of A.

$$\lambda_+ = 1 + 2i$$

$$A - \lambda_+ I = \begin{bmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{bmatrix}$$

$$A - \lambda_+ I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \xleftarrow{\cdot\, i} \to \begin{bmatrix} 2 & 2i \\ -2 & -2i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\left.\begin{array}{l} v_1 = -i\, v_2 \\ v_2 = 1, \quad v_1 = -i \end{array}\right] \Rightarrow \boxed{\underline{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_+ = 1+2i}$$

<u>Recall</u>: $\underline{v}^{(-)} = \overline{\underline{v}^{(+)}} \Rightarrow \boxed{\underline{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \lambda_- = 1-2i}$

$$\left[\begin{array}{l} \underline{\text{Recall}}: \quad \underline{v}^{(\pm)} = \underline{a} \pm \underline{b}\, i \qquad \lambda_\pm = \alpha \pm \beta i \\[1em] \underline{x}^{(1)} = e^{\alpha t} \left(\underline{a} \cos(\beta t) - \underline{b} \sin(\beta t) \right) \\[1em] \underline{x}^{(2)} = e^{\alpha t} \left(\underline{a} \sin(\beta t) + \underline{b} \cos(\beta t) \right). \end{array}\right]$$

<u>Hint</u> <u>to</u> <u>remember</u> <u>formulas</u> <u>for</u> $\underline{x}^{(1)}, \underline{x}^{(2)}$:

$$\underline{x}^{(+)} = \underline{v}^{(+)} e^{\lambda_+ t}$$

$$\underline{x}^{(+)} = (\underline{a} + \underline{b}\, i) \left[\cos(\beta t) + i \sin(\beta t)\right] e^{\alpha t}$$

$$\underline{x}^{(+)} = \underbrace{\left[\underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)\right] e^{\alpha t}}_{\underline{x}^{(1)}} + i \underbrace{\left[\underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)\right] e^{\alpha t}}_{\underline{x}^{(2)}}$$

back to our problem:

$$\underline{v}^{\pm} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$$

$$\boxed{\underline{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \quad \boxed{\underline{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

$$\lambda_\pm = 1 \pm 2i$$

$$\boxed{\alpha = 1}$$

$$\boxed{\beta = 2}$$

$$\underline{x}^{(1)} = e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(2t) \right)$$

$$\boxed{\underline{x}^{(1)} = e^{t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}$$

$$\underline{x}^{(2)} = e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(2t) \right)$$

$$\boxed{\underline{x}^{(2)} = e^{t} \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}}$$

$$\boxed{\underline{x}(t) = \left(c_1 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix} \right) e^{t}}$$

* Exam November 12, 2008

(Prblm 3)

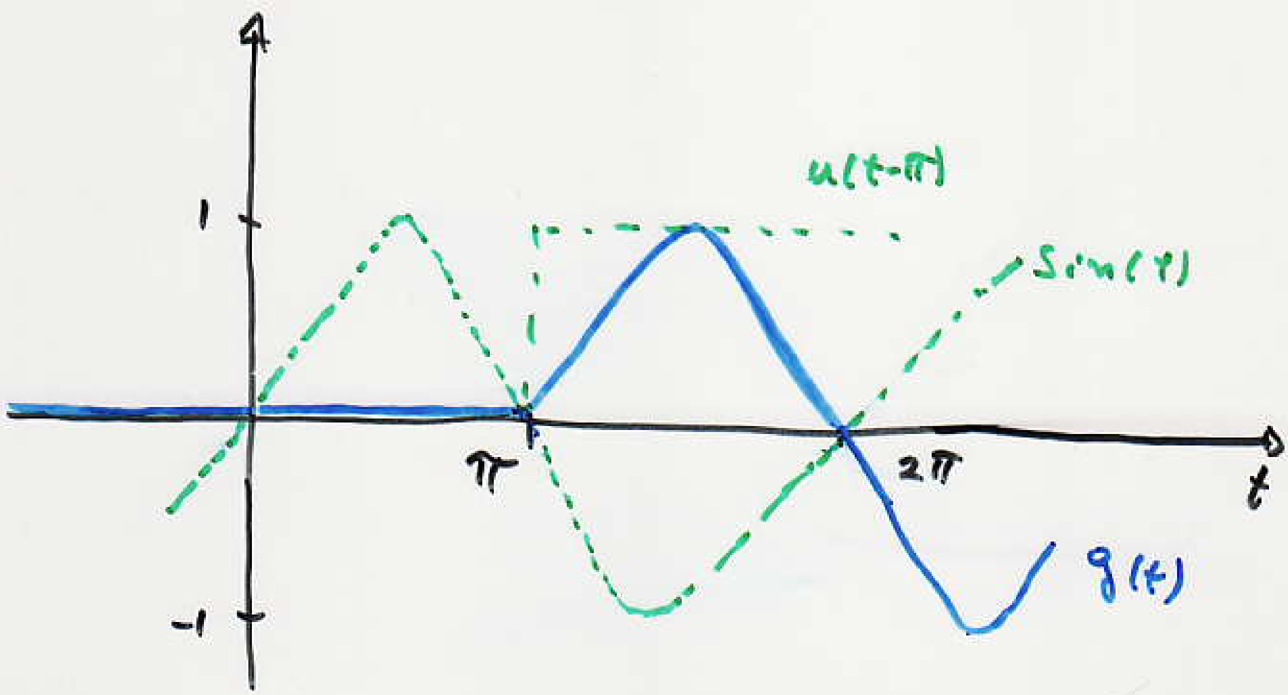
(Too long)

Sketch the graph of g and use L.T. to find y sol. of

$$y'' - 6y = g(t), \quad y(0) = y'(0) = 0$$

$$g(t) = \begin{cases} 0 & t < \pi \\ \sin(t-\pi) & t \geq \pi \end{cases}$$

Sol.



- Express g in terms of step functions.

$$\boxed{g(t) = u(t-\pi)\ \sin(t-\pi)}$$

$$\mathcal{L}[g] = e^{-\pi s}\ \mathcal{L}[\sin(t)]$$

$$\boxed{\mathcal{L}[g] = \frac{e^{-\pi s}}{s^2+1}}$$

$$\mathcal{L}[y''] - 6\ \mathcal{L}[y] = \mathcal{L}[g]$$

I.C. : $y(0) = y'(0) = 0$

$$(s^2-6)\ \mathcal{L}[y] = \frac{e^{-\pi s}}{s^2+1}$$

$$\boxed{\mathcal{L}[y] = e^{-\pi s}\ \frac{1}{(s^2+1)\ (s^2-6)}}$$

$$\boxed{(s^2-6) = (s+\sqrt{6})(s-\sqrt{6})}$$

$$\boxed{\mathcal{L}[y] = e^{-\pi s}\frac{1}{(s-\sqrt{6})(s+\sqrt{6})(s^2+1)}}$$

Partial fractions.

$$\frac{1}{(s+\sqrt{6})(s-\sqrt{6})(s^2+1)} = \frac{a}{(s+\sqrt{6})} + \frac{b}{(s-\sqrt{6})} + \frac{(cs+d)}{(s^2+1)}$$

$$\boxed{\begin{aligned} 1 = \; & a(s-\sqrt{6})(s^2+1) + b(s+\sqrt{6})(s^2+1) \\ & + (cs+d)(s^2-6) \end{aligned}}$$

$$\begin{aligned} 1 = \; & a(s^3 - \sqrt{6}\,s^2 + s - \sqrt{6}) \\ & + b(s^3 + \sqrt{6}\,s^2 + s + \sqrt{6}) \\ & + c\,s^3 - 6cs \quad + d s^2 - 6d \end{aligned}$$

$$\boxed{\begin{aligned} 1 = {} & (a+b+c)\, s^3 + \left[(-a+b)\sqrt{6} + d\right] s^2 \\ & + (a+b-6c)\, s + \left[(-a+b)\sqrt{6} - 6d\right] \end{aligned}}$$

$$a+b+c = 0$$

$$(-a+b)\sqrt{6} + d = 0$$

$$a+b-6c = 0$$

$$(-a+b)\sqrt{6} - 6d = 1$$

$$\Rightarrow c + 6c = 0 \Rightarrow \boxed{c = 0}$$

$$\Rightarrow d + 6d = -1 \Rightarrow \boxed{d = -\frac{1}{7}}$$

$$a+b = 0 \Rightarrow b = -a$$

$$(-a+b)\sqrt{6} = \frac{1}{7}$$

$$\Rightarrow (-a-a)\sqrt{6} = \frac{1}{7}$$

$$\boxed{a = -\frac{1}{14\sqrt{6}}}$$

$$\boxed{b = \frac{1}{14\sqrt{6}}}$$

$$\frac{1}{(s+\sqrt{6})(s-\sqrt{6})(s^2+1)} = -\frac{1}{14\sqrt{6}}\,\frac{1}{(s+\sqrt{6})}$$

$$+\frac{1}{14\sqrt{6}}\,\frac{1}{(s-\sqrt{6})} - \frac{1}{7}\,\frac{1}{(s^2+1)}$$

$$= \frac{1}{7}\Big[-\frac{1}{2\sqrt{6}}\,\mathcal{L}[e^{-\sqrt{6}t}] + \frac{1}{2\sqrt{6}}\,\mathcal{L}[e^{\sqrt{6}t}]$$

$$- \mathcal{L}[\sin(t)]\Big]$$

$$h(t) = \frac{1}{7}\Big[-\frac{1}{2\sqrt{6}}\,e^{-\sqrt{6}t} + \frac{1}{2\sqrt{6}}\,e^{\sqrt{6}t}$$

$$- \sin(t)\Big]$$

$$\boxed{h(t) = \frac{1}{14\sqrt{6}}\left(e^{-\sqrt{6}t} + e^{\sqrt{6}t} - 2\sqrt{6}\,\sin(t)\right)}$$

$$\boxed{\frac{1}{(s+\sqrt{6})(s-\sqrt{6})(s^2+1)} = \mathcal{L}[h(t)]}$$

$$\mathcal{L}[y] = e^{-\pi s}\frac{1}{(s+\sqrt{6})(s-\sqrt{6})(s^2+1)}$$

$$\mathcal{L}[y] = e^{-\pi s}\,\mathcal{L}[h(t)]$$

$$= \mathcal{L}[u(t-\pi)\,h(t-\pi)]$$

$$\boxed{y(t) = u(t-\pi)\,h(t-\pi)}$$

$$\boxed{h(t) = \frac{1}{14\sqrt{6}}\left(e^{-\sqrt{6}t} + e^{\sqrt{6}t} - 2\sqrt{6}\sin(t)\right)}$$