mth 235 L34

Plan: * Two-point BVP

* Comparison: IVP - BVP

* Existence, uniqueness of solutions to BVP.

* Particular case of BVP: Eigenvalue - Eigenfunction problems.

(10.1)

Exam 3: 6.1 - 6.6, 7.1 - 7.6, 7.8.

* Two-point Boundary Value Problem (BVP)

Def: A two-point BVP is the following:
Given functions p, q, g,
constants:

$$x_1 < x_2, \quad y_1, y_2$$

$$b_1, b_2 \quad (\text{not both zero})$$

$$\tilde{b}_1, \tilde{b}_2 \quad (\text{not both zero}),$$

find a function y sol. of

$$\boxed{y'' + p(x)\, y' + q(x)\, y = g(x)} \qquad (1)$$

$$\boxed{\begin{aligned} b_1\, y(x_1) + b_2\, y'(x_1) &= y_1 \\ \tilde{b}_1\, y(x_2) + \tilde{b}_2\, y'(x_2) &= y_2 \end{aligned}} \qquad (2)$$

Notation: Eqs in (2) are called boundary conditions (BC)

Remark: We only study constant coeff eqs.:

$$\boxed{y'' + a_1 y' + a_0 y = g} \qquad (3)$$

* **Particular cases of BVP.**

(1) Find y sol. of :
$$\left[\begin{array}{ll} y'' + a_1 y' + a_0 y = g(x) & \\ y(x_1) = Y_1 & \\ y(x_2) = Y_2 & \boxed{x_1 < x_2} \end{array}\right]$$

(2) Find y sol. of :
$$\left[\begin{array}{ll} y'' + a_1 y' + a_0 y = g(x) & \\ y'(x_1) = Y_1 & \\ y'(x_2) = Y_2 & \boxed{x_1 < x_2} \end{array}\right]$$

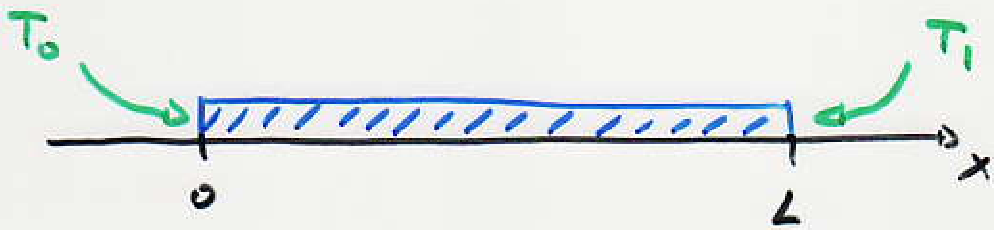
(3) Find y sol. of :
$$\left[\begin{array}{ll} y'' + a_1 y' + a_0 y = g(x) & \\ y(x_1) = Y_1 & \\ y'(x_2) = Y_2 & \boxed{x_1 \neq x_2} \end{array}\right]$$

* **Example from physics:**

The equilibrium (time independent) temperature of a thin bar of length L with the bar extremes kept at temperatures T_0, T_1 is the solution T of the BVP:

$$\boxed{T''(x) = 0} \qquad x \in (0, L)$$

$$\boxed{\begin{aligned} T(0) &= T_0 \\ T(L) &= T_1 \end{aligned}} \qquad \text{BC}$$



* **Comparison : IVP - BVP.**

Diff. eq: $$\boxed{y'' + a_1 y' + a_0 y = 0} \qquad (3)$$

(homogeneous eq.)

IVP : [Find $y(t)$ sol. of (3) satisfying

IC : $$\boxed{\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_1 \end{aligned}}$$]

In physics: $y(t)$: position, t : time

IC: position, velocity both at $t=0$.
(initial conditions)

BVP:

$$\left[\begin{array}{l} \text{Find } y(x) \text{ sol of (3) satisfying.} \\ \text{BC: } \boxed{\begin{array}{l} y(0) = Y_0 \\ y(L) = Y_1 \end{array}} \end{array}\right]$$

<u>In physics</u>:

$y(x)$: physical quantity (temperature)

x : position

B.C : conditions at the boundary of the object we study.

* Existence and Uniqueness of solutions

Thrm: (IVP) Consider the IVP:

$$\begin{bmatrix} y'' + a_1 y' + a_0 y = 0 \\ y(0) = Y_0, \quad y'(0) = Y_1, \end{bmatrix} \quad (4)$$

and let $r_\pm$ be the roots of

$$\left[p(r) = r^2 + a_1 r + a_0 \right]$$

If $r_+ \neq r_-$, real or complex,

then, for every choice of Y_0, Y_1 there exists a unique sol. y to the IVP in (4).

Summary: The IVP in (4) always has a unique sol. no matter what Y_0 and Y_1 we choose.

Thrm: (BVP) Consider the BVP:

$$\left[\begin{array}{l} y'' + a_1 y' + a_0 y = 0 \\ y(0) = y_0 \quad , \quad y(L) = y_1 \;, \end{array}\right] \quad (5)$$

and let $r_\pm$ be the roots of

$$\left[\, p(r) = r^2 + a_1 r + a_0 . \,\right]$$

(A) - If $r_+ \neq r_-$, real, then for every choice of $L \neq 0, y_0, y_1$ there exist a unique sol. y to the BVP in (5).

(B) - If $r_\pm = \alpha \pm \beta i$, $\beta \neq 0$, then the solutions to (5) fall into one of these possibilities:

(1) There exists a unique sol.

(2) There exists NO solution.

(3) There exist infinitely many solutions.

Proof Thrm IVP:

General sol. of $y'' + a_1 y' + a_0 y = 0$ is:

$$y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t}$$

$$y'(t) = c_1 r_+ e^{r_+ t} + c_2 r_- e^{r_- t}$$

IC: $\left.\begin{array}{l} y_0 = y(0) = c_1 + c_2 \\ y_1 = y'(0) = c_1 r_+ + c_2 r_- \end{array}\right] \Rightarrow$

$$\begin{bmatrix} 1 & 1 \\ r_+ & r_- \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$X = \begin{bmatrix} 1 & 1 \\ r_+ & r_- \end{bmatrix}$ is invertibe, since

$\boxed{\det(X) = r_- - r_+ \neq 0}$ ← important step.

$\underline{c} = X^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ always exists for every choice of y_0, y_1.

∎

Proof BVP case A.

General sol. of $y'' + a_1 y' + a_0 y = 0$ is

$$y(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}$$

BC:
$$y_0 = y(0) = c_1 + c_2$$
$$y_1 = y(L) = c_1 e^{r_+ L} + c_2 e^{r_- L}$$

$$\begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$X = \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix}$ is invertible, since

$$\boxed{\det(X) = e^{r_- L} - e^{r_+ L} \neq 0}$$ $r_+ \neq r_-$ (real)

$\underline{c} = X^{-1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ always exists for every choice of y_0, y_1 and $L \neq 0$

∎

<u>Proof</u> <u>BVP</u> <u>case</u> <u>B</u>: (The important case)

The general sol. of $y'' + a_1 y' + a_0 y = 0$ is

$$y(x) = c_1 e^{r_+ x} + c_2 e^{r_- x}$$

where $r_\pm = \alpha \pm \beta i$, $\beta \neq 0$.

BC: $y_0 = y(0) = c_1 + c_2$

$$y_1 = y(L) = c_1 e^{r_+ L} + c_2 e^{r_- L}$$

$$\begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 1 \\ e^{r_+ L} & e^{r_- L} \end{bmatrix}$$ may not be invertible.

$$\det(X) = e^{r_- L} - e^{r_+ L}$$

$$= e^{\alpha L} \left(e^{-i\beta L} - e^{i\beta L} \right)$$

$$\det(X) = e^{-\alpha L}(-2i)\,\sin(\beta L)$$

so $\boxed{\det(X) = 0}$ iff $\boxed{\beta L = n\pi}$

n: integer

Therefore, the following possibilities arise:

If $\beta L \neq n\pi$, Then BVP has a unique sol.

If $\beta L = n\pi$, Then BVP can have either: (1) No solution

or (2) Infinitely many sols.

∎

<u>Example</u> : Find y sol. BVP

$$y'' + y = 0 \ ,$$

$$y(0) = 1 \ , \qquad y(\pi) = -1 \ .$$

<u>Sol</u>:

$$p(r) = r^2 + 1 \quad \Rightarrow \quad \boxed{r_{\pm} = \pm i}$$

The general sol. is

$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$

BC:

$$\left.\begin{array}{l} 1 = y(0) = c_1 \\ -1 = y(\pi) = -c_1 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} c_1 = 1 \\ c_2 : \text{ free.} \end{array}}$$

$$\boxed{y(x) = \cos(x) + c_2 \sin(x)} \qquad c_2 \in \mathbb{R}$$

$\boxed{\text{infinitely many sols.}}$

Example: Find y sol. BVP

$$y'' + y = 0$$

$$y(0) = 1, \quad y(\pi) = 0$$

Sol:

$$P(r) = r^2 + 1 \Rightarrow r_{\pm} = \pm i.$$

The general sol. is

$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$

BC:

$$\left.\begin{aligned} 1 &= y(0) = c_1 \\ 0 &= y(\pi) = -c_1 \end{aligned}\right] \Rightarrow \boxed{\text{No solution}}$$

Example: Find y sol. BVP

$$y'' + y = 0$$

$$y(0) = 1 \quad , \quad y\left(\frac{\pi}{2}\right) = 1 .$$

Sol:

$$p(r) = r^2 + 1 \quad \Rightarrow \quad r_{\pm} = \pm i$$

The general sol. is

$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$

BC:

$$\left.\begin{aligned} 1 &= y(0) = c_1 \\ 1 &= y\left(\frac{\pi}{2}\right) = c_2 \end{aligned}\right] \Rightarrow \boxed{c_1 = c_2 = 1.}$$

$$\boxed{y(x) = \cos(x) + \sin(x)}$$

$\boxed{\text{unique solution.}}$

The Eigenvalue - Eigenfunction Problem

Problem: Find a number λ and a non-zero function y solutions of the BVP

$$\boxed{y''(x) + \lambda\, y(x) = 0} \qquad (\mathrm{I})$$

$$\boxed{y(0) = 0, \quad y(L) = 0} \qquad (\mathrm{II})$$

Remark: Similar to the eigenvalue-eigenvector problem: Given $n \times n$ A, find λ, non-zero $\underline{v}$ sol. of

$$A\underline{v} - \lambda \underline{v} = \underline{0}$$

$A \to$ second derivative.

$\underline{v} \to$ a function y.

Thrm:

If $\lambda \leq 0$,

Then (I)-(II) has No solution.

If $\lambda > 0$,

Then, there exist infinitely many eigenvalues λ_n and eigenfunctions Y_n given by

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$$

$$\boxed{Y_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$$

with n positive integer.

Proof :

case $\lambda \leq 0$: Let $\lambda = -\mu^2$

with $\mu \geq 0$.

$\boxed{\text{case } \mu = 0}$ trivial: The problem is:

$$Y''(x) = 0, \qquad Y(0) = 0, \qquad Y(L) = 0$$

The general sol. is: $Y(x) = c_1 + c_2 x$

$$\left.\begin{aligned} 0 &= Y(0) = c_1 \quad \Rightarrow \boxed{c_1 = 0} \\ 0 &= Y(L) = c_1 + c_2 L \end{aligned}\right] \Rightarrow \boxed{c_2 = 0}$$

so $\boxed{Y(x) = 0}$. $\boxed{\text{There is NO non-zero sol.}}$

$\boxed{\text{case } \mu > 0}$, $\lambda = -\mu^2$

$$P(r) = r^2 - \mu^2 = 0 \Rightarrow \boxed{r_\pm = \pm \mu}$$

The general sol. is:

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} .$$

BC: $0 = y(0) = c_1 + c_2$

$0 = y(L) = c_1 e^{\mu L} + c_2 e^{-\mu L}$

$$\begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$X = \begin{bmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{bmatrix}$ is invertible, since

$\det(X) = e^{-\mu L} - e^{\mu L} \neq 0 \Rightarrow \boxed{\underline{c} = \underline{0}}$

$\boxed{y(x) = 0}$: $\boxed{\text{There is NO non-zero solution.}}$

case $\lambda > 0$: Let $\boxed{\lambda = \mu^2}$, $\mu > 0$.

$P(r) = r^2 + \mu^2 \Rightarrow \boxed{r_\pm = \pm \mu i}$

General sol: $\boxed{Y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)}$

BC: $0 = Y(0) = c_1$

$0 = Y(L) = c_1 \cos(\mu L) + c_2 \sin(\mu L)$ $\Rightarrow$

$\Rightarrow \boxed{c_1 = 0} \Rightarrow c_2 \sin(\mu L) = 0$

We look for non-zero $Y(x)$, so $c_2 \neq 0$.

So: $\boxed{\sin(\mu L) = 0}$ $\Rightarrow$

equation for μ, so for λ.

$\Rightarrow \mu L = n\pi \quad n \geq 1 \Rightarrow \boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$

$Y(x) = c_n \sin\left(\frac{n\pi}{L}x\right)$

choosing $c_n = 1$ $\Rightarrow \boxed{Y_n(x) = \sin\left(\frac{n\pi}{L}x\right)}$