mth 235 L 33

Plan: * Homogeneous, constant coeff. $n \times n$ linear systems.

* Repeated eigenvalues.

* 2×2 systems : phase portraits.

(7.8)

* Review: The solutions to $\underline{x}'(t) = A\,\underline{x}(t)$ depend on the eigen-pairs of A.

$n \times n$ A

n distinct eigenvalues

$\Downarrow$

l.i. set of n eigenvectors

Real case (7.5)

complex case (7.6)

repeated eigenvalues

l.i. set of n eigenvectors

(A)

$\boxed{\text{l.i. set of } m < n \text{ eigenvectors}}$

(B)

(7.8)

* Review :

Def : [The eigenvalue λ of an $n \times n$ matrix A is called repeated iff λ has algebraic multiplicity $r > 1$.]

* Recall. [If $\{\lambda_1, \dots, \lambda_k\}$ are (complex) eigenvalues of $n \times n$ matrix A, $k \leq n$, then

$$P(\lambda) = (\lambda - \lambda_1)^{r_1} \dots (\lambda - \lambda_k)^{r_k}$$

and r_i is the algebraic multiplicity of λ_i , $i = 1, \dots, k$.]

[λ_i repeated iff $r_i > 1$.]

* Review: $\left[\text{ } n\times n \text{ matrices with repeated eigenvalues have l.i. sets of } m \leq n \text{ eigenvectors.} \right]$

* Example:

(1) $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

has eigenvalues: $\boxed{\begin{array}{lll} \lambda_1 = 3, & r_1 = 2, & s_1 = 2 \\ \lambda_2 = 1, & r_2 = 1, & s_2 = 1 \end{array}}$

(s_1, s_2: geometric multiplicities)

and a l.i. set of $m = 3 = n$ eigenvectors

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}}$$

(2) $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

has eigenvalues : $\boxed{\begin{matrix} \lambda_1 = 3 , & \underline{r_1 = 2} , & \underline{s_1 = 1} \\ \lambda_2 = 1 , & r_2 = 1 , & s_2 = 1 \end{matrix}}$

and a l.i. set of eigenvectors $m = 2 < 3 = n$

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} .}$$

*Remark : Fundamental sols. to $\underline{x}'(t) = A\,\underline{x}(t)$ are simple to find in the case that the $n \times n$ matrix A has repeated eigenvalues and a l.i. set of n eigenvectors.

* Example : Find a fundamental set of sols. to

$$\underline{x}'(t) = A\,\underline{x}(t) , \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:

The eigen-pairs of A are:

$$\lambda_1 = 3 , \qquad \left\{ \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \; \underline{w}^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$\lambda_2 = 1 , \qquad \left\{ \underline{v}^{(2)} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$$

Then, a fundamental set is:

$$\boxed{\left\{ \begin{aligned} \underline{x}^{(1)}(t) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} , \\ \underline{x}^{(2)}(t) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t} , \\ \underline{x}^{(3)}(t) &= \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} e^{t} \end{aligned} \right\}}$$

* Main result: repeated eigenvalues with l.i. set of $m < n$ eigenvectors

Thm: If λ is an eigenvalue of the $n \times n$ matrix A having algebraic multiplicity $r = 2$, geometric multiplicity $s = 1$ and eigenvector $\underline{v}$, then a l.i. set with two solutions of

$$\boxed{\underline{x}'(t) = A\,\underline{x}(t)}$$

is given by

$$\boxed{\left\{ \underline{x}^{(1)}(t) = \underline{v}\,e^{\lambda t},\quad \underline{x}^{(2)}(t) = (\underline{v}\,t + \underline{w})\,e^{\lambda t} \right\}}$$

where the vector $\underline{w}$ is sol. of

$$\boxed{(A - \lambda I)\,\underline{w} = \underline{v}}\,.$$

sketch of the proof in text book.

* **Recall**: The case of a single, second order eq.

$$\boxed{y'' + a_1 y' + a_0 y = 0}$$

with characteristic polyn.

$$p(r) = r^2 + a_1 r + a_0 = (r - r_1)^2.$$

A fundamental set is:

$$\left\{ y_1(t) = e^{r_1 t}, \quad y_2(t) = t\, e^{r_1 t} \right\}$$

that is: $\boxed{y_2(t) = t\, y_1(t)}$

This is not the case with vector-valued eqs.

$$\boxed{\text{In general}: \quad \underline{W} \neq \underline{0}}$$

Example : Find a fundamental set of

$$x' = A x \quad , \quad A = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

Sol:

(1) Find eigen-pairs of A.

- Eigenvalues:

$$P(\lambda) = \begin{vmatrix} -\frac{3}{2} - \lambda & 1 \\ -\frac{1}{4} & -\frac{1}{2} - \lambda \end{vmatrix} = (\lambda + \tfrac{1}{2})(\lambda + \tfrac{3}{2}) + \tfrac{1}{4}$$

$$P(\lambda) = \lambda^2 + \tfrac{3}{2}\lambda + \tfrac{1}{2}\lambda + \tfrac{3}{4} + \tfrac{1}{4}$$

$$\boxed{P(\lambda) = \lambda^2 + 2\lambda + 1}$$

$$P(\lambda) = 0 \Rightarrow \lambda_{\pm} = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

$$\boxed{P(\lambda) = (\lambda + 1)^2} \qquad \boxed{\lambda_1 = -1 \ , \ r_1 = 2}$$

- Eigenvectors :

$\lambda_1 = -1$

$$A + I = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \quad v_1 = 2v_2 \quad \Rightarrow \quad \boxed{\underline{v}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \lambda_1 = -1}$$

$$\boxed{r_1 = 2, \quad s_1 = 1}$$

(2) Find the vector $\underline{w}$ sol. of

$$\boxed{(A - \lambda_1 I)\, \underline{w} = \underline{v}^{(1)}}$$

that is

$$(A + I)\, \underline{w} = \underline{v}^{(1)}$$

$$\boxed{\begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right] \begin{array}{c} \leftarrow -2 \\ \leftarrow -4 \end{array} \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 1 & -2 & -4 \end{array}\right] \to$$

$$\to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} w_1 = 2w_2 - 4 \\ 0 = 0 \end{array}$$

the system has sols.

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2w_2 - 4 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$\boxed{\underline{w} = \underline{v}^{(1)} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}} \qquad \underline{v}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Choose simplest sol.: $w_2 = 0 \Rightarrow \boxed{\underline{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}}$

A fundamental set is:

$$\boxed{\left\{ \underline{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \quad \underline{x}^{(2)}(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t} \right\}}$$

Example: Find sol. to IVP

$$\underline{x}'(t) = A\,\underline{x}(t), \qquad A = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Sol:

The general sol. is

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}}$$

Initial condition:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{x}(0) = c_1 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\underline{v}^{(1)}} + c_2 \underbrace{\begin{bmatrix} -4 \\ 0 \end{bmatrix}}_{\underline{w}}$$

$$\boxed{\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{0+4}\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 \\ 1 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}}$$

$$\boxed{\underline{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{4}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) e^{-t}}$$

Also:

$$\boxed{\underline{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\left(1 + \frac{t}{4}\right) e^{-t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-t}}$$

← useful for phase portraits

Also:

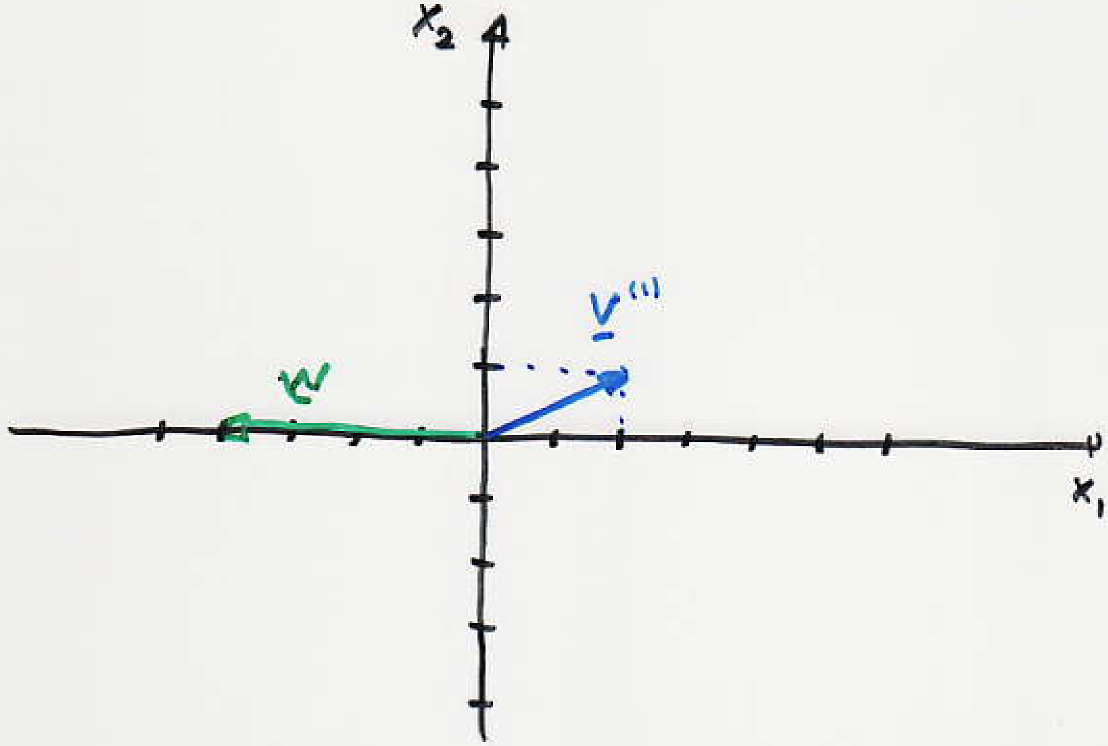
$$\boxed{\underline{x}(t) = \begin{bmatrix} \left(1 + \frac{t}{2}\right) e^{-t} \\ \left(1 + \frac{t}{4}\right) e^{-t} \end{bmatrix}}$$

← useful for components graphs.

* The phase diagram of previous examples. (2x2)

(1) plot vectors $\underline{v}^{(1)}$, $\underline{w}$.

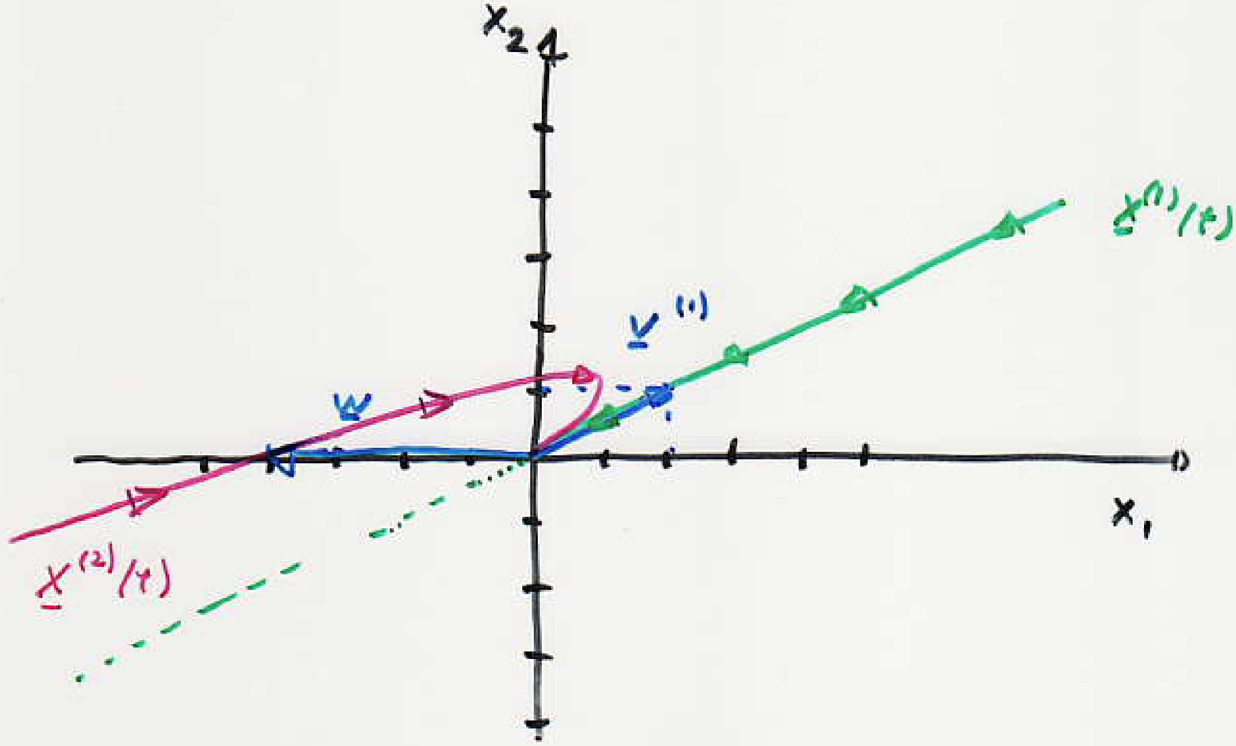
$$\underline{v}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix},$$



(2) plot solutions $\underline{x}^{(1)} = \underline{v}\, e^{\lambda t}$

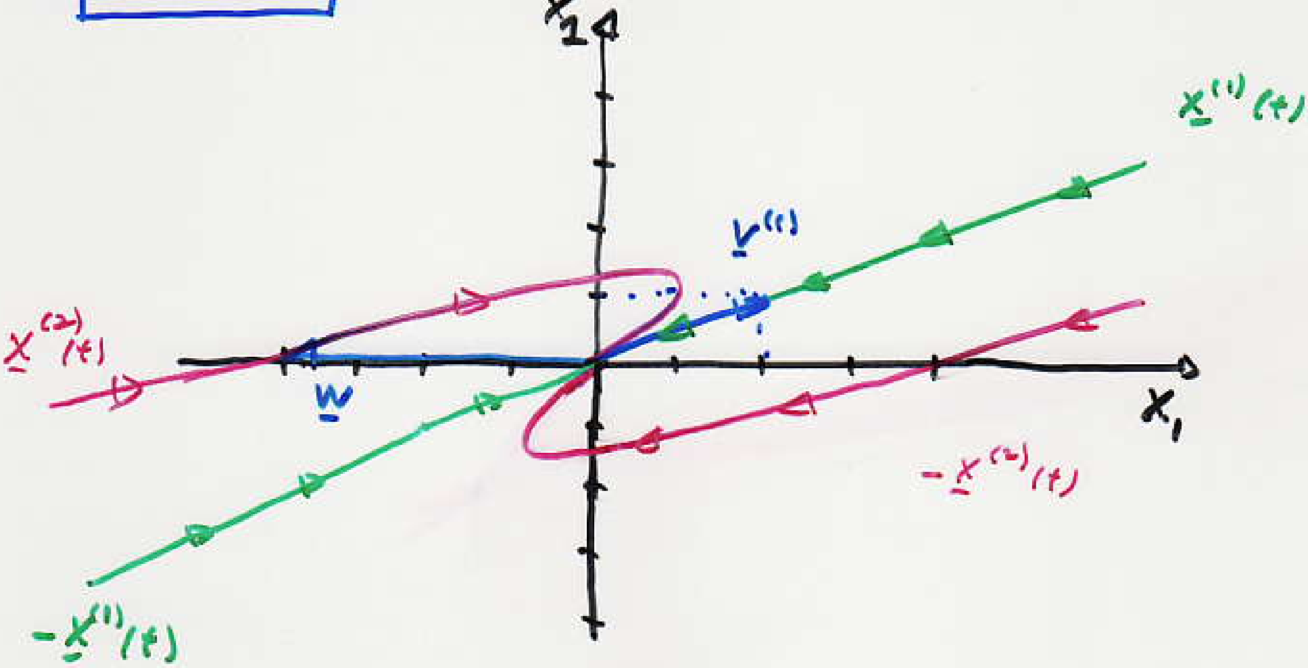
$$\underline{x}^{(2)} = (\underline{v}\, t + \underline{w})\, e^{\lambda t}$$

our case $\lambda = -1$, $\boxed{\lambda < 0}$

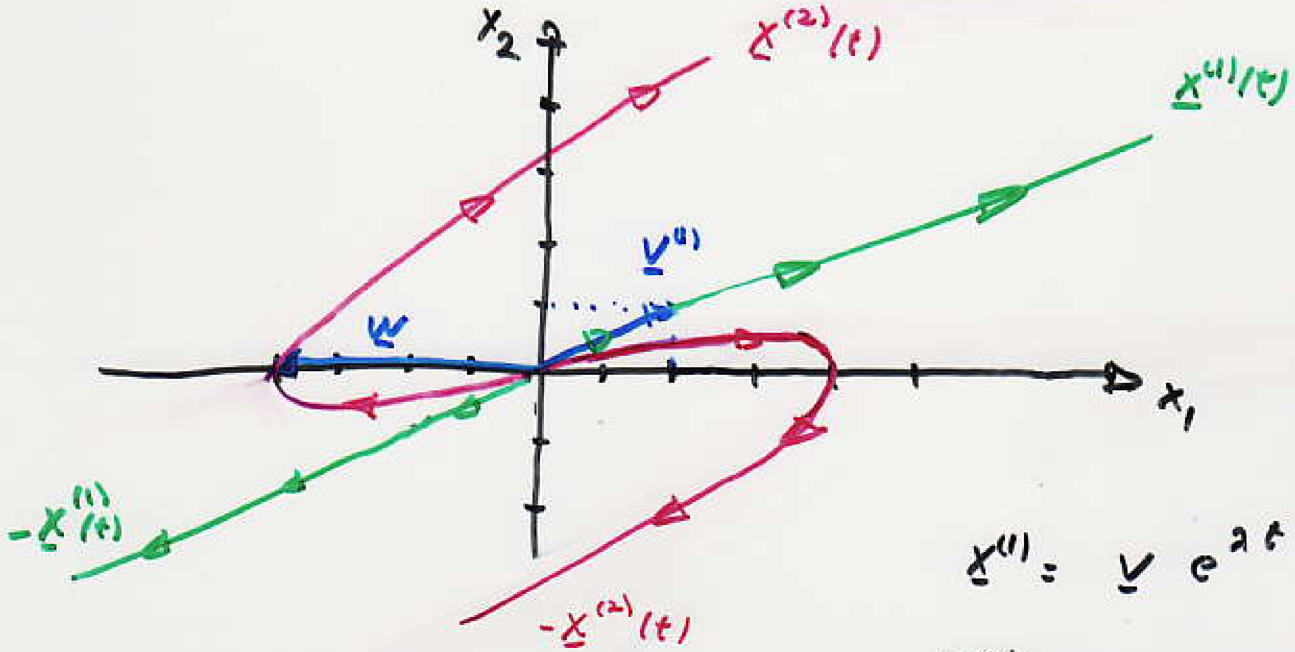


(3) plot the solutions : $\underline{x}^{(1)}, -\underline{x}^{(1)}, \underline{x}^{(2)}, -\underline{x}^{(2)}$.

$\boxed{\lambda < 0}$



* Example : case $\boxed{\lambda > 0}$ same $\underline{v}^{(1)}$, $\underline{w}$.



$\underline{x}^{(1)} = \underline{v}\, e^{\lambda t}$

$\underline{x}^{(2)} = (\underline{v}\, t + \underline{w})\, e^{\lambda t}$

* Example : [Plot the vector components of the sol. in the previous example]

sol:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(1+\frac{t}{2}\right) e^{-t} \\ \left(1+\frac{t}{4}\right) e^{-t} \end{bmatrix}$$