mth 235 L32

Plan:

* Homogeneous, constant coeff., $n \times n$ linear systems.
* n distinct complex-valued eigenvalues.
* 2×2 systems: phase portraits.

(7.6)

* **Review**: The solution to $\underline{x}'(t) = A\,\underline{x}(t)$ depends on the eigen-pairs of A.

$n \times n$ A

- n distinct eigenvalues
  - l.i. set of n eigenvalues
    - Real case (7.5)
    - complex case (7.6)
- repeated eigenvalues
  - l.i. set of n eigenvalues
  - l.i. set of $m < n$ eigenvalues
  - (7.8)

$\uparrow$

* Review: Main Result.

Thrm: If $n \times n$ matrix A has a l.i. set of n eigenvectors $\{ \underline{v}^{(1)}, \dots, \underline{v}^{(n)} \}$. with corresponding eigenvalues $\{ \lambda_1, \dots, \lambda_n \}$, then a fundamental set of solutions to the differential system

$$\boxed{\underline{x}'(t) = A \, \underline{x}(t)}$$

is given by

$$\boxed{\left\{ \underline{x}^{(1)}(t) = \underline{v}^{(1)} e^{\lambda_1 t}, \dots, \underline{x}^{(n)}(t) = \underline{v}^{(n)} e^{\lambda_n t} \right\}}$$

* Real-valued matrix $\underline{A}$ with complex eigenvalues.

Propos. [If $\{\lambda, \underline{v}\}$ is an eigen-pair of an $n\times n$ real-valued matrix A, then $\{\bar{\lambda}, \bar{\underline{v}}\}$ also is an eigen-pair of matrix A.]

Proof: By hypothesis

$$\boxed{A\underline{v} = \lambda \underline{v}} \quad \text{and} \quad \bar{A} = A,$$

then

$$\overline{A\underline{v}} = \overline{\lambda \underline{v}} \quad \Leftrightarrow \quad \bar{A}\,\bar{\underline{v}} = \bar{\lambda}\,\bar{\underline{v}}$$

so

$$\boxed{A\bar{\underline{v}} = \bar{\lambda}\,\bar{\underline{v}}}$$

∎

Remark: [If $\boxed{\lambda_1 = \alpha + \beta i}$, $\boxed{\underline{v}^{(1)} = \underline{a} + \underline{b}\, i}$ is an eigen-pair of A real, with $\alpha, \beta \in \mathbb{R}$, $\underline{a}, \underline{b} \in \mathbb{R}^n$, then so is

$$\boxed{\lambda_2 = \alpha - \beta i}, \quad \boxed{\underline{v}^{(2)} = \underline{a} - \underline{b}\, i}$$]

Thrm: If an $n \times n$ real-valued matrix A has eigen-pairs

$$\boxed{\lambda_\pm = \alpha \pm \beta i}, \quad \boxed{\underline{v}^{(\pm)} = \underline{a} \pm \underline{b} i,}$$

with $\alpha, \beta \in \mathbb{R}$, $\underline{a}, \underline{b} \in \mathbb{R}^n$, then a l.i. set of two real-valued solutions to the diff. eq.

$$\boxed{\underline{x}'(t) = A\, \underline{x}(t)}$$

is given by the elements

$$\boxed{\underline{x}^{(1)}(t) = e^{\alpha t} \left[\underline{a} \cos(\beta t) - \underline{b} \sin(\beta t) \right]}$$

$$\boxed{\underline{x}^{(2)}(t) = e^{\alpha t} \left[\underline{a} \sin(\beta t) + \underline{b} \cos(\beta t) \right]}$$

Remarks : (1) $\lambda_- = \bar{\lambda}_+$

$$\underline{v}^{(-)} = \overline{\underline{v}^{(+)}}$$

(2) $\{\lambda_+, \underline{v}^{(+)}\}$, $\{\lambda_-, \underline{v}^{(-)}\}$

eigen-pairs of matrix A,
so, two solutions of

$$\underline{x}'(t) = A\,\underline{x}(t)$$

are given by

$$\left\{ \boxed{\begin{aligned} \underline{x}^{(+)}(t) &= \underline{v}^{(+)} e^{\lambda_+ t} \\ \underline{x}^{(-)}(t) &= \underline{v}^{(-)} e^{\lambda_- t} \end{aligned}} \right\} \text{ l.i.}$$

These solutions are complex-valued.

Proof of Thrm:

$$\underline{x}^{(+)} = (\underline{a} + \underline{b}i)\, e^{(\alpha + \beta i)t}$$

$$= (\underline{a} + \underline{b}i)\, e^{\beta i t}\, e^{\alpha t}$$

$$= e^{\alpha t}\, (\underline{a} + \underline{b}i)\, \left(\cos(\beta t) + i \sin(\beta t)\right)$$

$$\boxed{\begin{aligned} \underline{x}^{(+)} &= e^{\alpha t}\left(\underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)\right) \\ &\quad + e^{\alpha t}\left(\underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)\right) i \end{aligned}}$$

$$\boxed{\begin{aligned} \underline{x}^{(-)} &= e^{\alpha t}\left(\underline{a}\cos(\beta t) - \underline{b}\sin(\beta t)\right) \\ &\quad - e^{\alpha t}\left(\underline{a}\sin(\beta t) + \underline{b}\cos(\beta t)\right) i \end{aligned}}$$

$$\boxed{\begin{aligned} \underline{x}^{(+)} &= \underline{x}^{(1)} + \underline{x}^{(2)} i \\ \underline{x}^{(-)} &= \underline{x}^{(1)} - \underline{x}^{(2)} i \end{aligned}}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \underline{x}^{(1)} &= \frac{1}{2}\left(\underline{x}^{(+)} + \underline{x}^{(-)}\right) \\ \underline{x}^{(2)} &= \frac{1}{2i}\left(\underline{x}^{(+)} - \underline{x}^{(-)}\right) \end{aligned}} \Rightarrow \boxed{\begin{array}{l} \underline{x}^{(1)}, \underline{x}^{(2)} \text{ are sols.} \\ \text{to } \quad \underline{x}' = A\underline{x} \\ \text{and } \{\underline{x}^{(1)}, \underline{x}^{(2)}\} \text{ l.i.} \end{array}}$$

$\blacksquare$

<u>Example</u> Find a real-valued fundamental set to

$$\underline{x}'(t) = A\,\underline{x}(t) \quad , \quad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

<u>Sol</u>:

(1) Find eigenvalues:

$$P(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9$$

$$0 = P(\lambda) = (\lambda-2)^2 + 9 \Rightarrow (\lambda_\pm - 2) = \pm\sqrt{-9}$$

$$\boxed{\lambda_\pm = 2 \pm 3i}$$

$$\lambda_\pm = \alpha \pm \beta i \Rightarrow \boxed{\alpha = 2}, \quad \boxed{\beta = 3}$$

(2) Find eigenvectors:

$$\lambda_+ = 2 + 3i$$

$$A - \lambda_+ I = \begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$$

$$A - \lambda_+ I = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \xrightarrow{\cdot i} \begin{bmatrix} 3 & 3i \\ -3 & -3i \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{v_1 = -v_2\, i}$$

$$\underline{v}^{(+)} = \begin{bmatrix} -v_2\, i \\ v_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} v_2$$

$$v_2 = 1 \Rightarrow \boxed{\underline{v}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix}} \quad \boxed{\lambda_+ = 2 + 3i}$$

$$\left(v_2 = i \Rightarrow \underline{v}^{(+)} = \begin{bmatrix} 1 \\ i \end{bmatrix} \right)$$

Now $\underline{v}^{(-)}$ is simple to obtain:

$$\underline{v}^{(-)} = \overline{\underline{v}^{(+)}} \Rightarrow \boxed{\underline{v}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix}}, \boxed{\lambda_- = 2 - 3i}$$

Notice: $\underline{v}^{(\pm)} = \begin{bmatrix} \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$

So: $\boxed{\underline{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$, $\boxed{\underline{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$

So, complex-valued fundamental sols. are:

$$\boxed{\underset{\sim}{x}^{(+)} = \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(2+3i)t}}$$

$$\boxed{\underset{\sim}{x}^{(-)} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(2-3i)t}}$$

Real-valued fundamental sols. are:

$$\underset{\sim}{x}^{(1)} = e^{2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \right)$$

$$\boxed{\underset{\sim}{x}^{(1)} = e^{2t} \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix}}$$

$$\underset{\sim}{x}^{(2)} = e^{2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \right)$$

$$\boxed{\underset{\sim}{x}^{(2)} = e^{2t} \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix}}$$

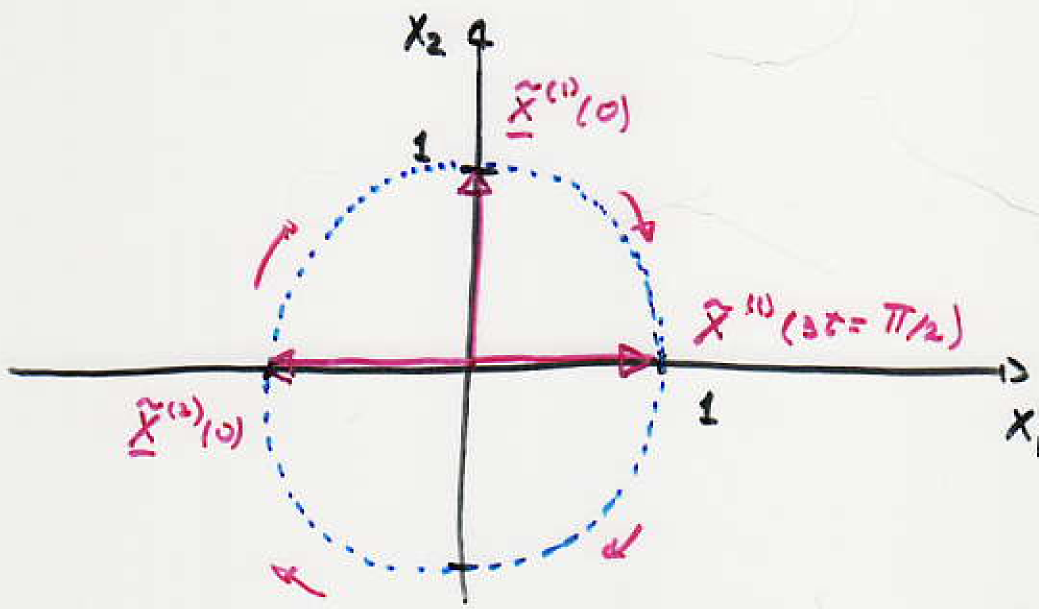
Example: [Sketch a qualitative phase portrait of the real-valued solution above.]

Sol:

$$\underline{x}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}, \qquad \underline{x}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}$$

(1) Sketch the closed curve associated with

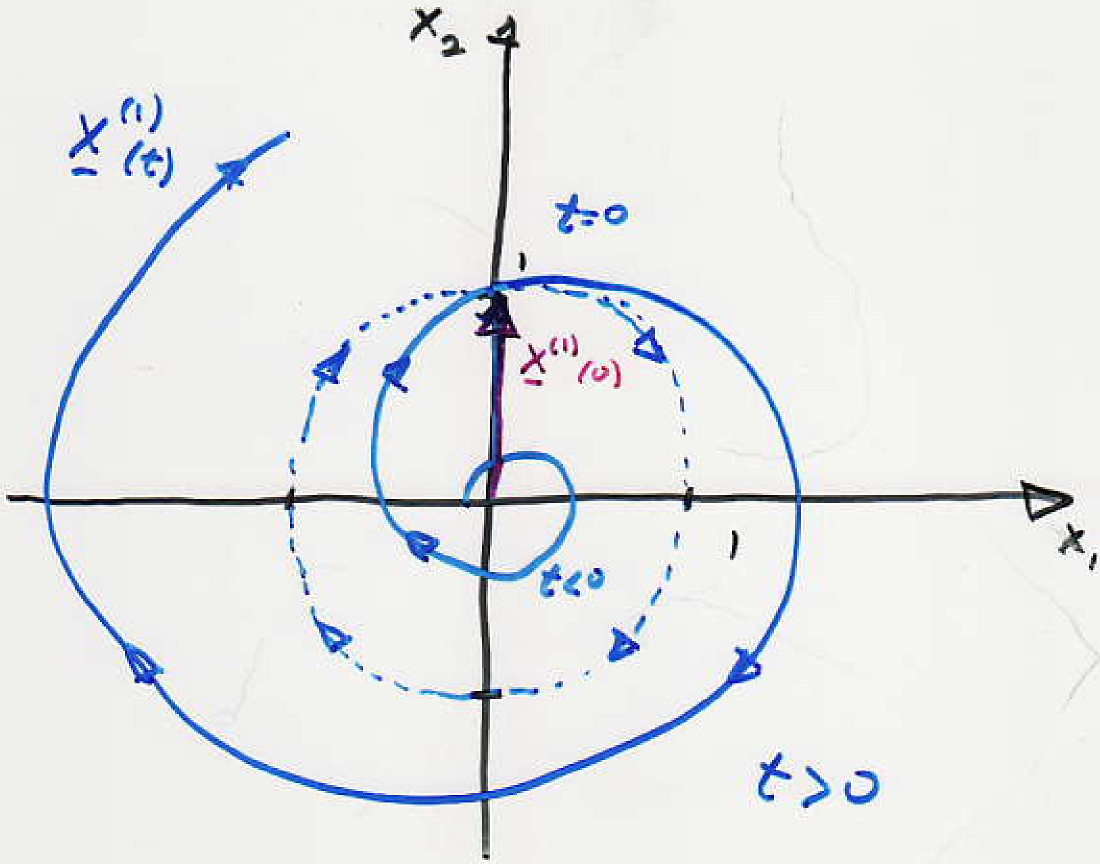
$$\tilde{\underline{x}}^{(1)} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix}, \qquad \tilde{\underline{x}}^{(2)} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix}.$$



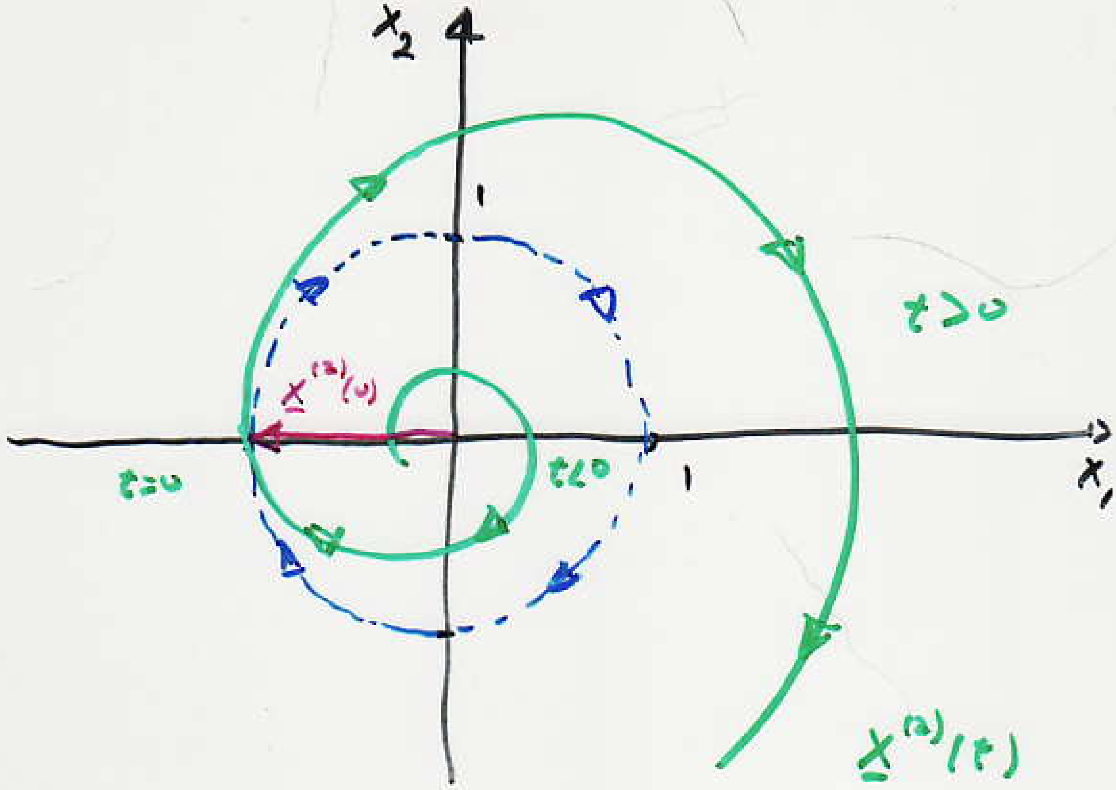
(2) Introduce the information from the exponential factor:

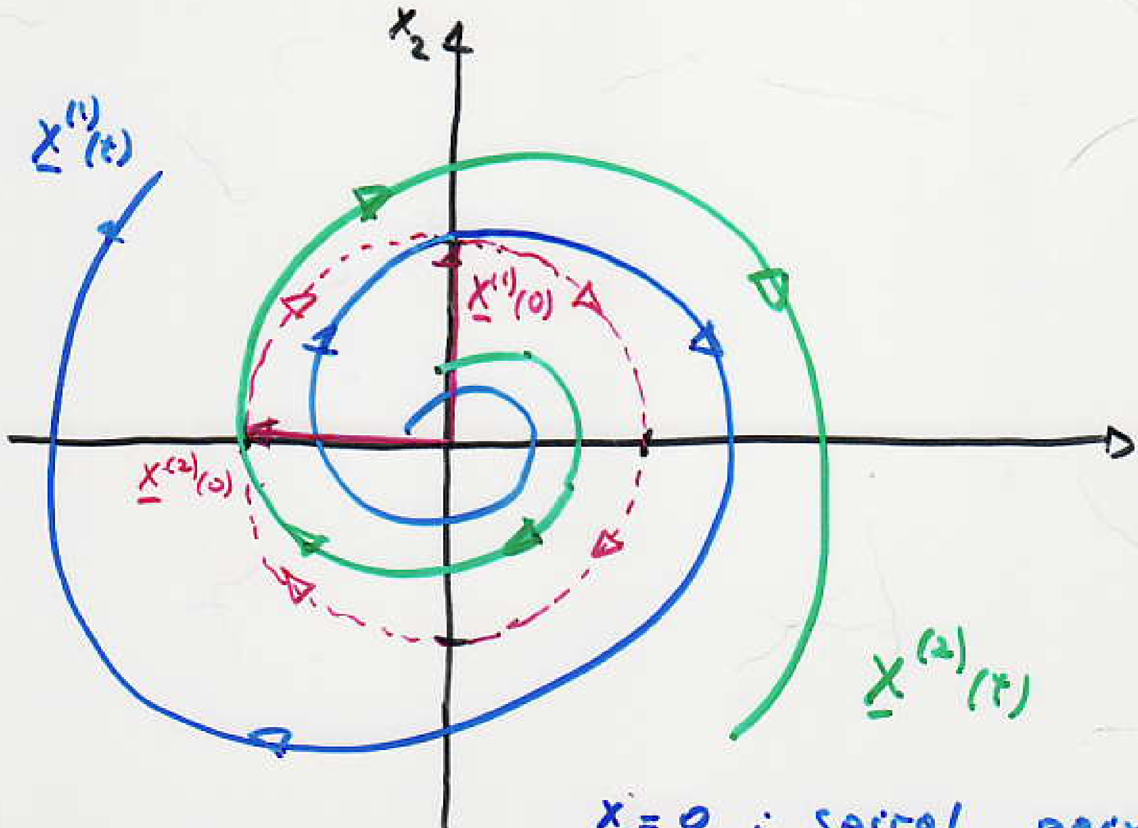
$$\boxed{2 = \alpha > 0}$$

$$\underset{\sim}{x}^{(1)}(t) = \underset{\sim}{\tilde{x}}^{(1)}(t) \quad e^{2t}$$



$$\underset{\sim}{x}^{(2)}(t) = \hat{\underset{\sim}{x}}^{(2)}(t) \; e^{2t}$$





$\underset{\sim}{x} = \underset{\sim}{0}$: spiral point.

Example: If 2×2 matrix A has eigen-pairs

$$\lambda_{\pm} = -1 \pm 2i, \quad \underline{v}^{(\pm)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \pm \begin{bmatrix} 3 \\ 1 \end{bmatrix} i,$$

then sketch a phase portrait of a real-valued fundamental set to

$$\underline{x}' = A\,\underline{x}.$$

Sol.

$$\lambda_{\pm} = \alpha \pm \beta i = -1 \pm 2i \Rightarrow \boxed{\alpha = -1}, \quad \boxed{\beta = 2}$$

$$\underline{v}^{(\pm)} = \underline{a} \pm \underline{b}\, i = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \pm \begin{bmatrix} 3 \\ 1 \end{bmatrix} i \Rightarrow \boxed{\underline{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \quad \boxed{\underline{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

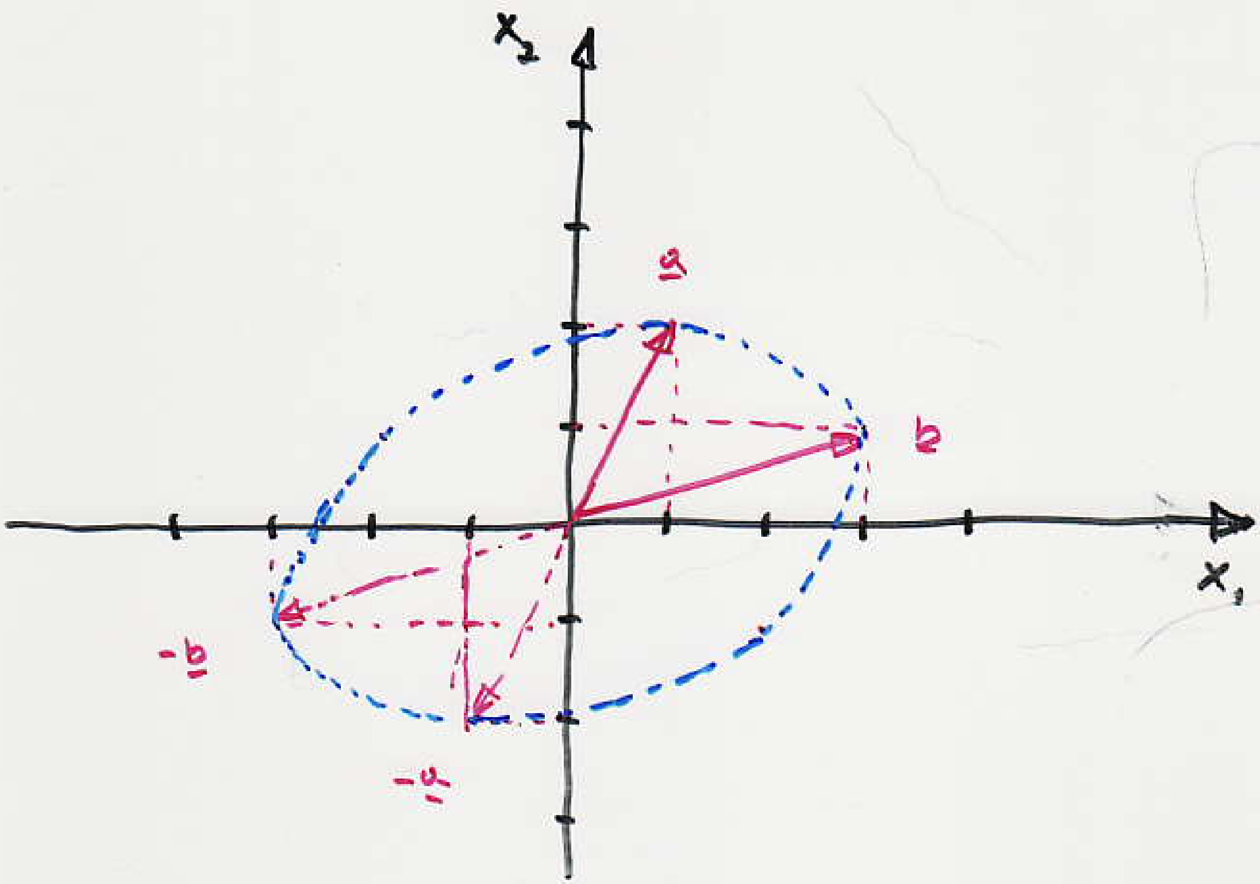
$$\boxed{\underline{x}^{(1)} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(2t) - \begin{bmatrix} 3 \\ 1 \end{bmatrix} \sin(2t) \right) e^{-t}}$$

$$\boxed{\underline{x}^{(2)} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(2t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cos(2t) \right) e^{-t}.}$$

(1) First sketch the closed curves

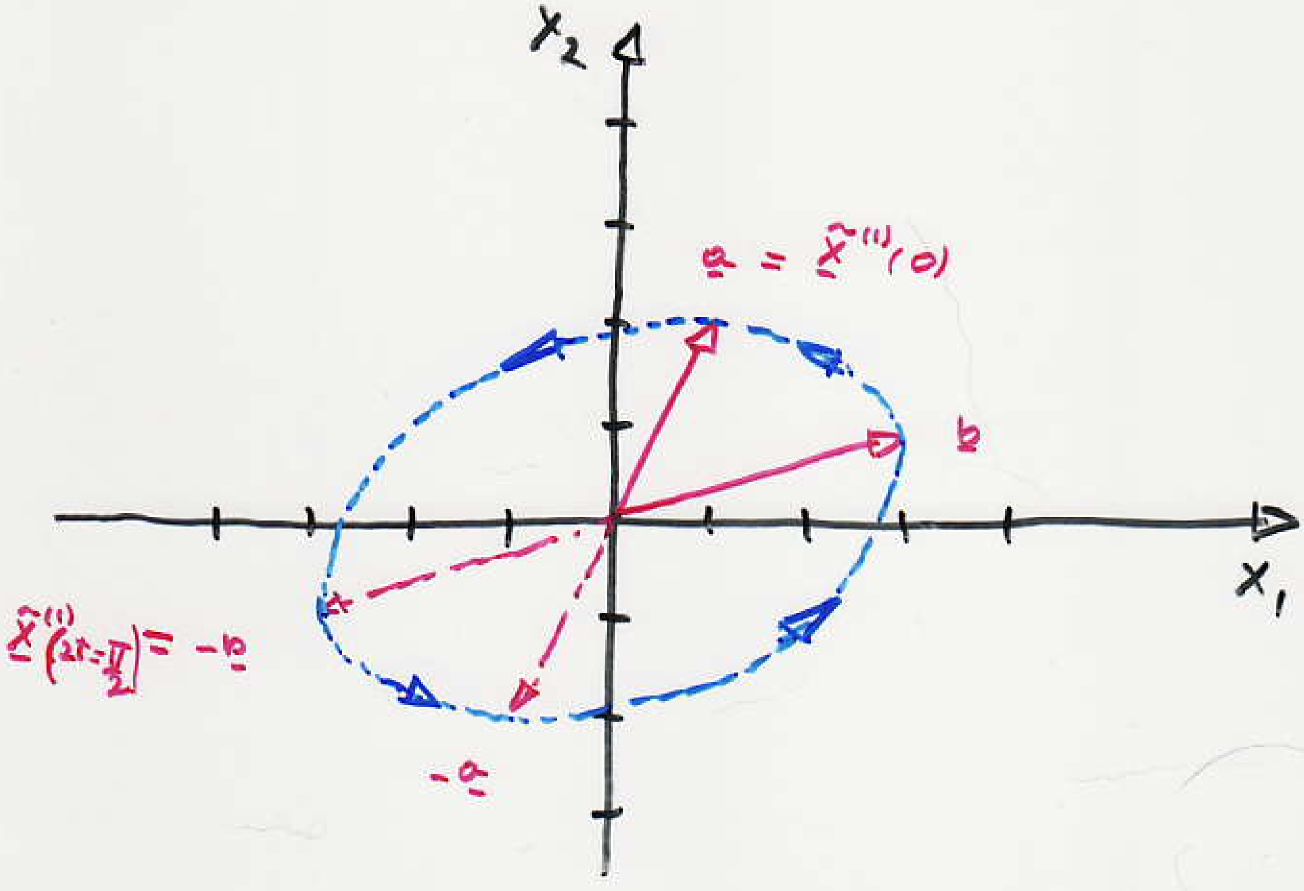
$$\underset{\sim}{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(2t) - \begin{bmatrix} 3 \\ 1 \end{bmatrix} \sin(2t)$$

$$\underset{\sim}{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(2t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cos(2t).$$



$$2t = 0 \Rightarrow \tilde{\underline{x}}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{a}$$

$$2t = \frac{\pi}{2} \Rightarrow \tilde{\underline{x}}^{(1)} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\underline{b}$$

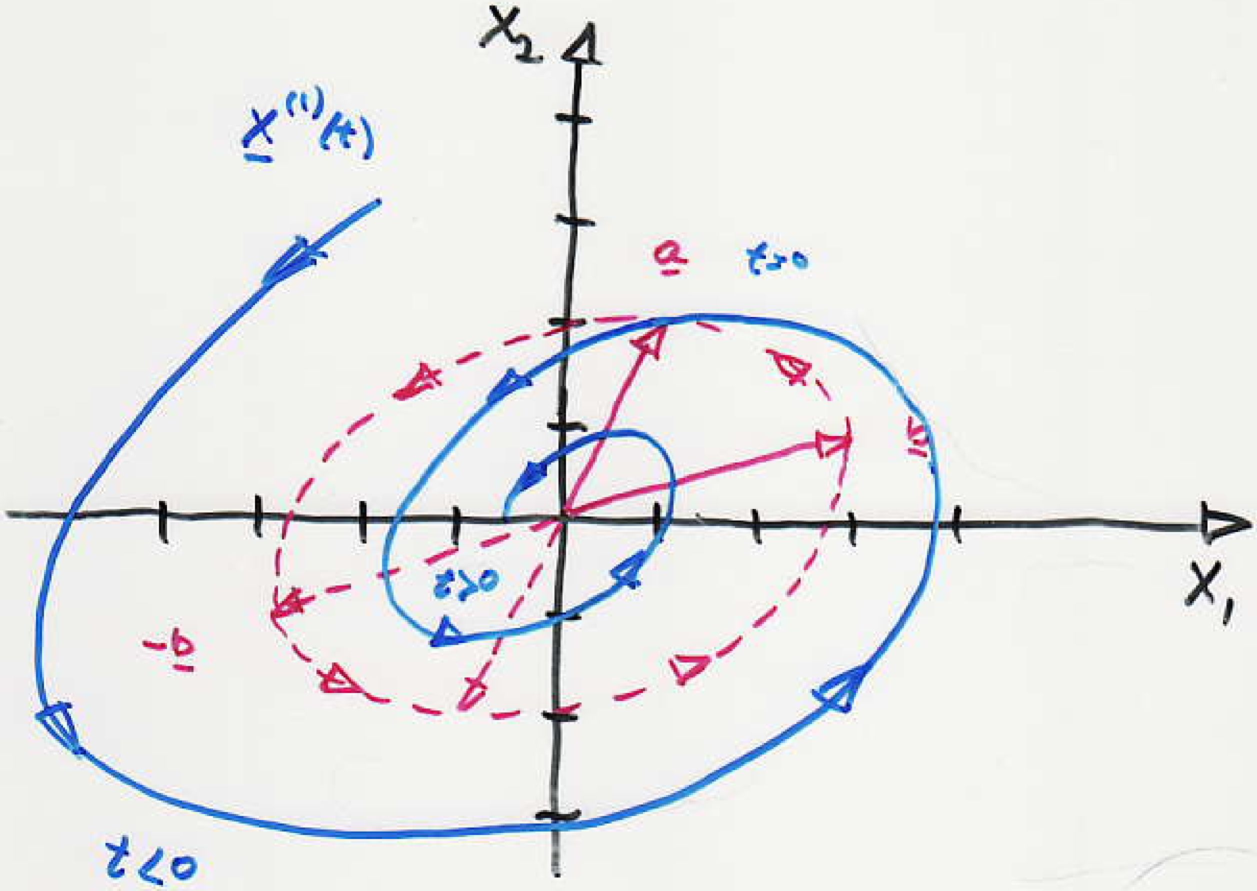


$$2t = 0 \Rightarrow \tilde{\underline{x}}^{(2)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \underline{b}$$

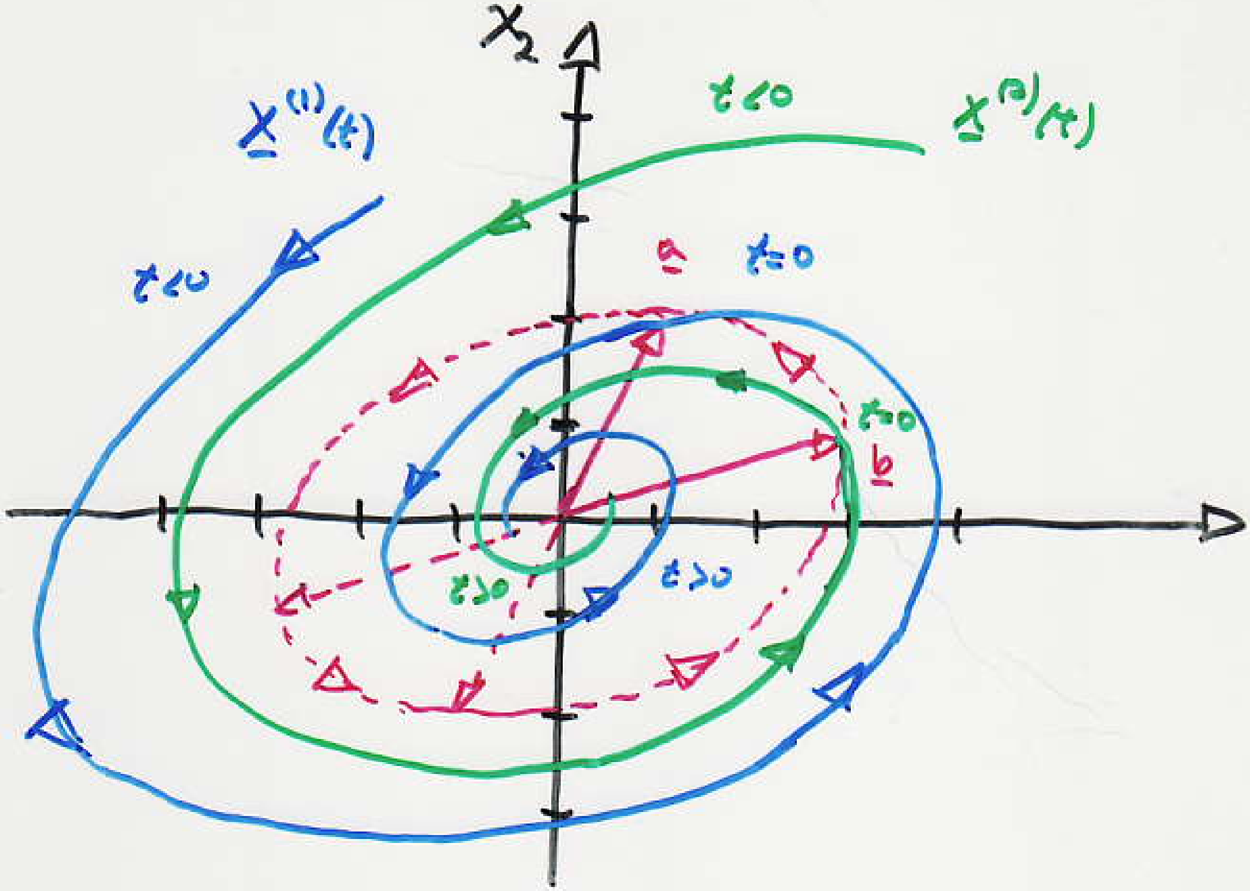
$$2t = \frac{\pi}{2} \Rightarrow \tilde{\underline{x}}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{a}$$

(2) Introduce the information from the exponential factor:

$$\boxed{-1 = \alpha < 0}$$



As t grows the solution vector spirals towards the origin.


x_2
$\underline{x}^{(1)}(t)$
$\underline{x}^{(2)}(t)$
$t<0$
$t<0$
$\underline{a}$
$t=0$
$t=0$
$\underline{b}$
$t>0$
$t>0$