mth 235 L31

Plan:

* $n \times n$ linear differential systems
* Homogeneous, constant coefficients systems
* Real, non-zero, distinct eigenvalues
* 2×2 systems: phase portraits.

(7.5)

* **Review:** $n \times n$ linear diff. systems

- Given an $n \times n$ $A(t)$, n-vector $\underline{b}(t)$, find $\underline{x}(t)$ sol. of

$$\boxed{\underline{x}'(t) = A(t)\, \underline{x}(t) + \underline{b}(t)} \qquad (1)$$

- The system (1) is homogeneous iff $\underline{b} = \underline{0}$, that is,

$$\boxed{\underline{x}'(t) = A(t)\, \underline{x}(t).} \qquad (2)$$

- The system (1) has constant coefficients iff matrix A does not depend on t, that is,

$$\boxed{\underline{x}'(t) = A\, \underline{x}(t) + \underline{b}(t)} \qquad (3)$$

- We study homogeneous, constant coeff. systems, that is,

$$\boxed{\underline{x}'(t) = A\, \underline{x}(t)} \qquad (4)$$

- Main result: (previous class)

[Given continuous functions A, $\underline{b}$ on $[0,T] \subset \mathbb{R}$, and $\underline{x}_0$, there exists a unique function $\underline{x}$ sol. of IVP

$$\underline{x}'(t) = A(t)\,\underline{x}(t) + \underline{b}(t)$$

$$\underline{x}(0) = \underline{x}_0$$]

- Today we learn how to find such solution in the case of $n\times n$, homogeneous, constant coefficients systems

$$\boxed{\underline{x}'(t) = A\,\underline{x}(t).}$$

* Eigen-pairs provide sols. to diff. systs..

Propos. If $\lambda, \underline{v}$ are eigenvalue and eigenvector of an $n\times n$ matrix A, then the n-vector-valued function

$$\boxed{\underline{x}(t) = \underline{v}\, e^{\lambda t}}\quad, \quad t \in \mathbb{R}$$

is a solution of the $n\times n$ linear homogeneous, constant coeff. diff. system

$$\boxed{\underline{x}'(t) = A\,\underline{x}(t)}.$$

Proof:

$$\underline{x}'(t) = \underline{v}\,(e^{\lambda t})'$$

$$= \lambda\, \underline{v}\, e^{\lambda t}$$

$$\boxed{\underline{x}'(t) = \lambda\, \underline{x}(t)}$$

$$A\,\underline{x}(t) = A\,\underline{v}\, e^{\lambda t}$$

$$= \lambda\, \underline{v}\, e^{\lambda t}$$

$$\boxed{A\,\underline{x}(t) = \lambda\, \underline{x}(t)}$$

so: $\boxed{\underline{x}'(t) = A\,\underline{x}(t)}$ ◻

Remarks :

- To find all solutions of $\underline{x}'(t) = A\,\underline{x}(t)$ is important to find all eigenvalues - eigenvectors of matrix A.

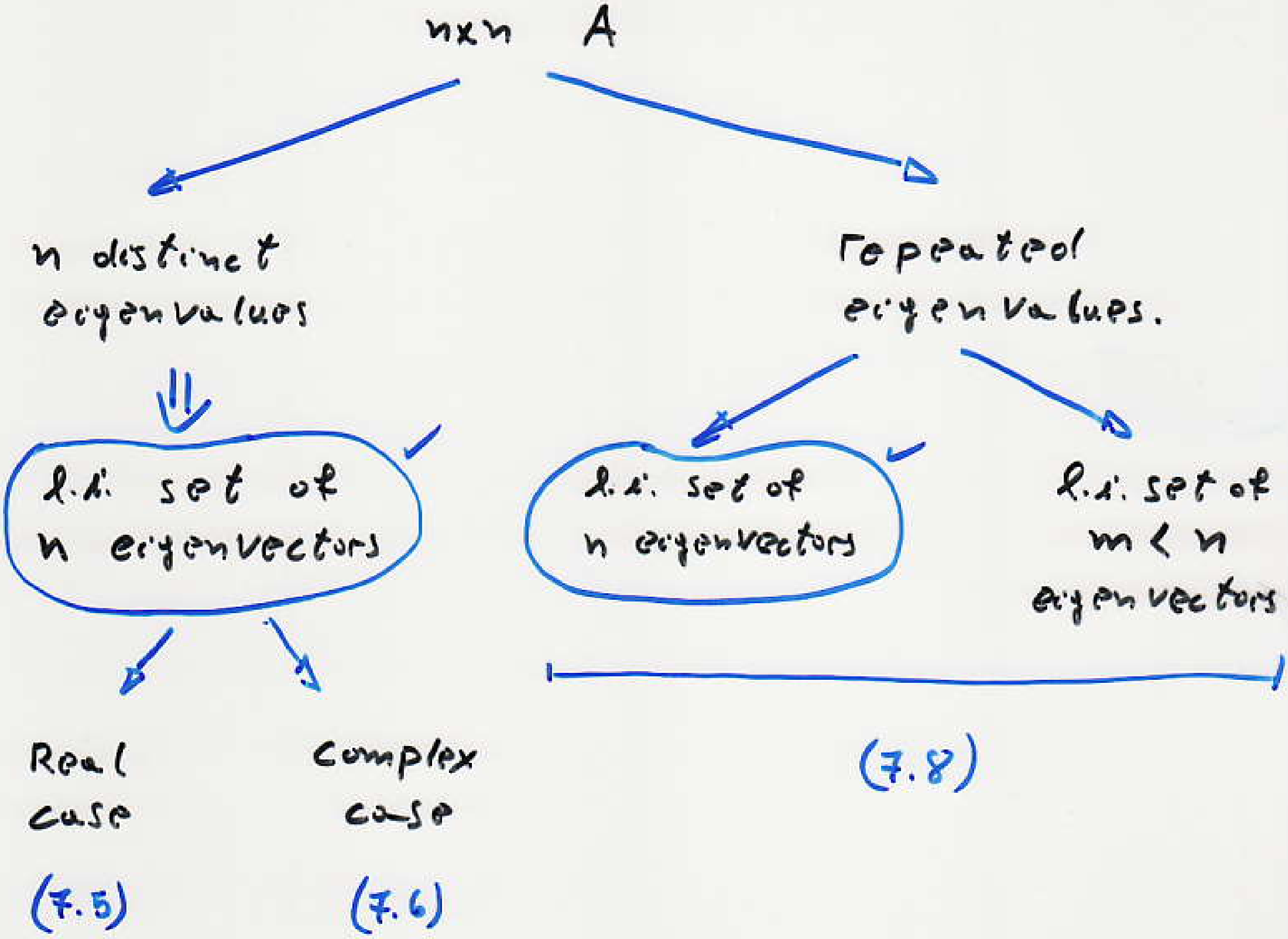
- Not every $n\times n$ matrix A has a l.i. set of n eigenvectors.

- We will study systems $\underline{x}'(t) = A\,\underline{x}(t)$ where $n\times n$ matrix A satisfies:

  (a) $\det(A) \neq 0$. (invertible).

  (b) A: real-valued matrix .

<u>Remark:</u> - From Linear Algebra we know that such $n \times n$ matrix must satisfy:



* $n \times n$ A having a l.i. set of n eigenvectors is important because:

Thrm: If $n \times n$ matrix A has a l.i. set of n eigenvectors $\{\underline{v}^{(1)}, \dots, \underline{v}^{(n)}\}$ with corresponding eigenvalues $\{\lambda_1, \dots, \lambda_n\}$,

Then, a fundamental set of solutions to the diff. system

$$\boxed{\underline{x}'(t) = A\,\underline{x}(t)} \qquad (4)$$

is given by

$$\boxed{\left\{ \underline{x}^{(1)}(t) = \underline{v}^{(1)} e^{\lambda_1 t}, \dots, \underline{x}^{(n)}(t) = \underline{v}^{(n)} e^{\lambda_n t} \right\}} \qquad (5)$$

Proof: $\underline{x}^{(i)} = \underline{v}^{(i)} e^{\lambda_i t}$ $\quad i = 1, \dots, n$ is sol. of (4).

Since:

$$\underline{x}^{(i)\prime}(t) = \lambda_i \underline{v}^{(i)} e^{\lambda_i t} = A \underline{v}^{(i)} e^{\lambda_i t} = A \underline{x}^{(i)}(t)$$

The set in (5) is a fundamental set, since

$$X(t) = [\, \underline{x}^{(1)}(t), \ldots, \underline{x}^{(n)}(t) \,]$$

has non-zero Wronskian

$$\boxed{W(t) = \det(X(t)) \neq 0.}$$

For the proof, recall Abel's Thm in this case:

[If $W(t_0) \neq 0$, $t_0 \in \mathbb{R}$, then $W(t) \neq 0$ for all $t \in \mathbb{R}$.]

In our case:

$$W(0) = \det(X(0))$$

$$= \det([\, \underline{v}^{(1)}, \ldots, \underline{v}^{(n)} \,]) \neq 0$$

since $\boxed{\{\underline{v}^{(1)}, \ldots, \underline{v}^{(n)}\} \text{ l.i.}}$

∎

* <u>Class</u> <u>Today</u>: We study the case:

- [$n \times n$ A has real, non-zero, n distinct eigenvalues $\lambda_1, \dots, \lambda_n$.]

- [From Linear Algebra: A has a l.i. set of n eigenvectors $\{ \underline{v}^{(1)}, \dots, \underline{v}^{(n)} \}$.]

- From Thrm above: $\boxed{\underline{x}'(t) = A\, \underline{x}(t)}$ has a fundamental set given by

$$\boxed{\begin{gathered} \left\{ \underline{x}^{(i)}(t) = \underline{v}^{(i)} e^{\lambda_i t} \right\} \\ i = 1, \dots, n. \end{gathered}}$$

* **Example**: Find the general sol. to

$$x'(t) = A\,x(t), \quad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Sol:

- Find all eigenvalues of A.

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda) - 9$$

$$(\lambda-1)^2 - 9 = 0 \quad \Rightarrow \quad \lambda_\pm - 1 = \pm 3$$

$$\lambda_\pm = 1 \pm 3$$

$$\boxed{\lambda_+ = 4}, \quad \boxed{\lambda_- = -2}$$

A 2x2 has two distinct eigenvalues.

so A has a l.i. set of two eigenvectors.

- Find all eigenvectors of A.

$\boxed{\lambda_+ = 4}$

$$A - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = v_2 \Rightarrow \boxed{v^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\boxed{\lambda_- = -2}$

$$A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \quad \Rightarrow \boxed{\underline{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

we verify that $\left\{ \underline{v}^{(+)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \underline{v}^{(-)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is l.i.

So, a fundamental sol. of $\underline{x}' = A\,\underline{x}$ is

$$\boxed{\left\{ \underline{x}^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}, \quad \underline{x}^{(2)}(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} \right\}.}$$

So the general sol. of $\underline{x}' = A\,\underline{x}$ is

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}}$$

in components: $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,

$$\boxed{\begin{aligned} x_1(t) &= c_1 e^{4t} - c_2 e^{-2t} \\ x_2(t) &= c_1 e^{4t} + c_2 e^{-2t} \end{aligned}}$$

$$\boxed{\lambda_{(-)} < 0 < \lambda_{(+)}}$$

Example : Find the sol. to the IVP

$$\underline{x}'(t) = A\,\underline{x}(t), \quad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Sol:

(1) Find the general sol. of $\underline{x}' = A\underline{x}$.

$$\boxed{\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.} \qquad (6)$$

(2) Impose the initial cond. on (6).

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \underline{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c_1 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} c_2$$

$$\boxed{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

$$X(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow [X(0)]^{-1} = \frac{1}{1+1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

$$\boxed{\underline{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}} \qquad (7)$$

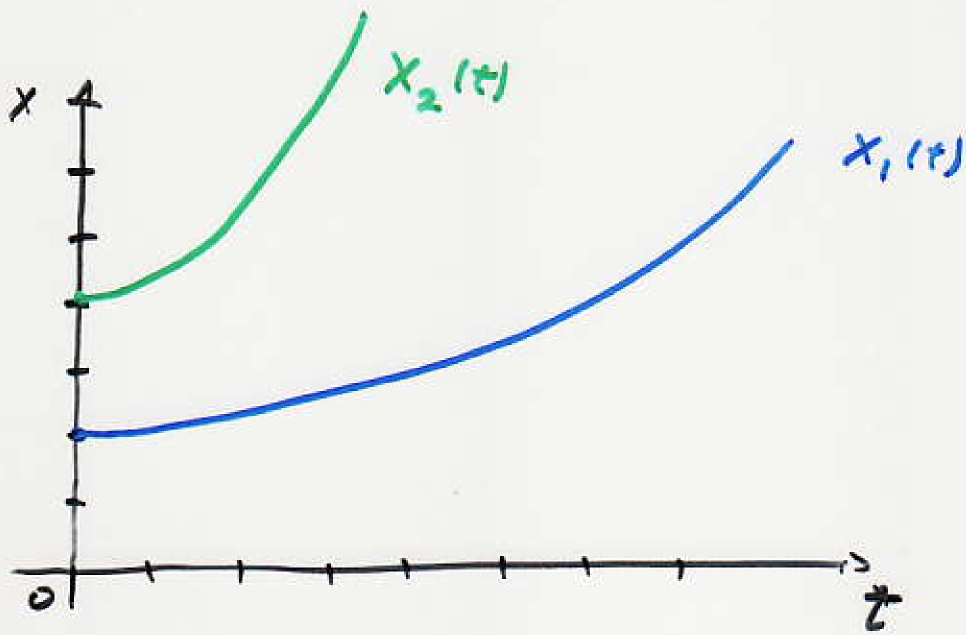
In components: $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,

$$\boxed{x_1(t) = 3e^{4t} - e^{-2t}} \qquad (8)$$

$$\boxed{x_2(t) = 3e^{4t} + e^{-2t}} \qquad (9)$$

* <u>Remarks</u>:

(1) We can always plot the graph of each component, $x_1(t)$, $x_2(t)$, as function of t.



(2) We can plot the whole vector $\underline{x}(t)$ on the plane.

This is called: $\boxed{\text{phase portrait or phase diagram.}}$

* Phase portraits : 2x2 systems.

- Let A be 2x2, with distinct, non-zero eigenvalues $\lambda_1 \neq \lambda_2$, and l.i. set $\{\underline{v}^{(1)}, \underline{v}^{(2)}\}$ of eigenvectors.

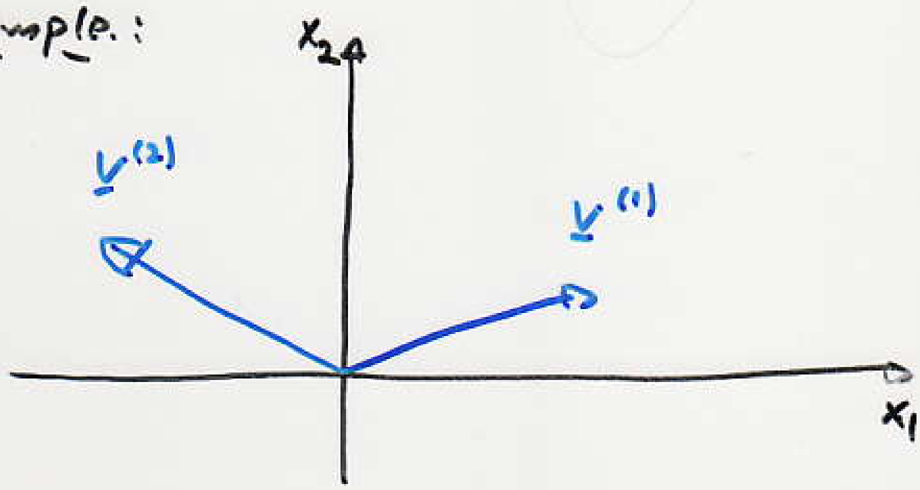
- So, the general sol. of $\underline{x}'(t) = A\,\underline{x}(t)$ is:

$$\boxed{\underline{x}(t) = c_1\,\underline{v}^{(1)}\,e^{\lambda_1 t} + c_2\,\underline{v}^{(2)}\,e^{\lambda_2 t}}$$

- Phase portrait: plot $\underline{x}(t)$ as function of t for all possible choices of c_1, c_2.

First: Plot $\underline{v}^{(1)}$, $\underline{v}^{(2)}$.

For example:



(No time in this plot.)

Second: Plot the simplest solutions:

$$\boxed{\begin{array}{l} \underline{x}^{(1)}(t) = \underline{v}^{(1)} e^{\lambda_1 t} \\ \hline \underline{x}^{(2)}(t) = \underline{v}^{(2)} e^{\lambda_2 t} \end{array}}$$

$(c_1 = 1,\ c_2 = 0)$

$(c_1 = 0,\ c_2 = 1)$

Third: Plot the general sol.

$$\boxed{\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + c_2 \underline{x}^{(2)}(t).}$$

—o—

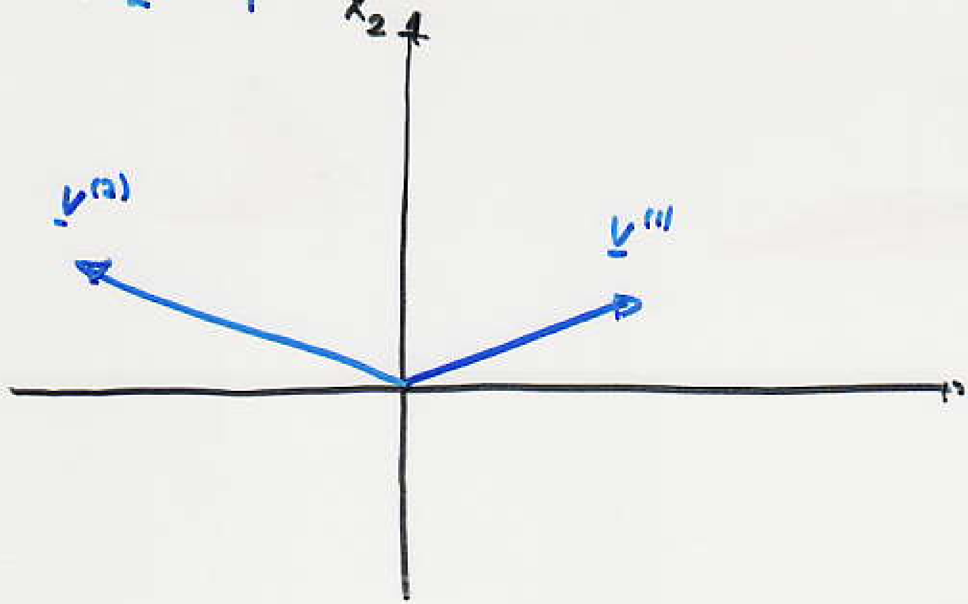
Back to step two: we need to study different cases:

(1) $\boxed{0 < \lambda_2 < \lambda_1}$

(2) $\boxed{\lambda_2 < 0 < \lambda_1}$

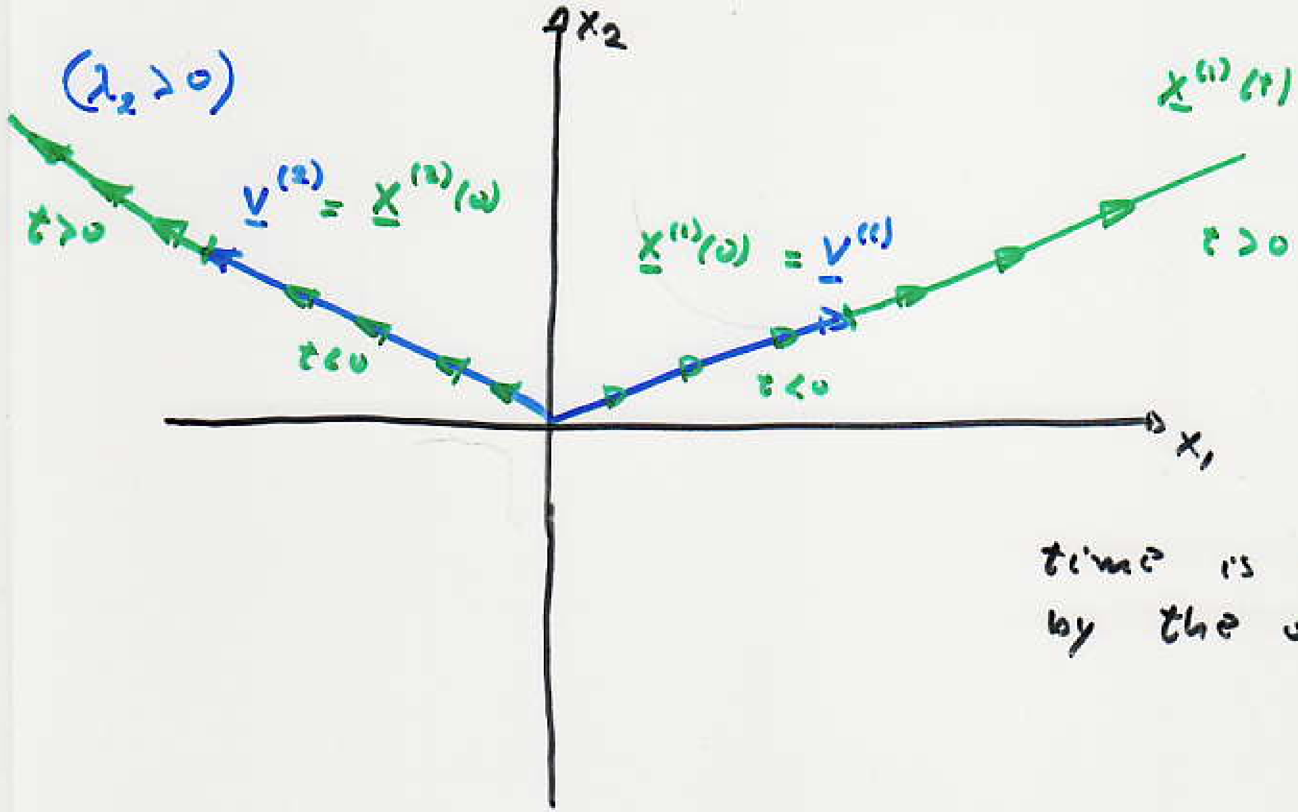
(3) $\boxed{\lambda_2 < \lambda_1 < 0}$

(1) $0 < \lambda_2 < \lambda_1$



(No time)

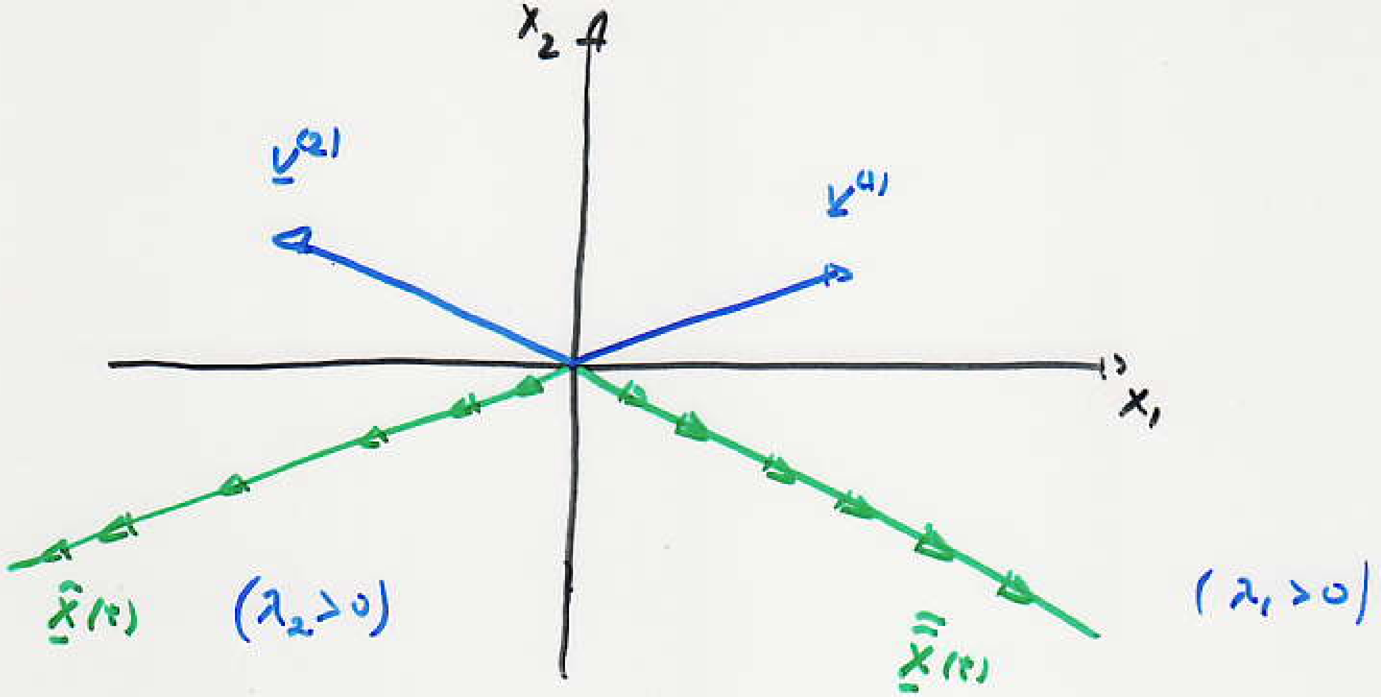
$$\underline{x}^{(1)}(t) = v^{(1)} e^{\lambda_1 t} \quad (c_1 = 1, c_2 = 0), \qquad \underline{x}^{(2)}(t) = v^{(2)} e^{\lambda_2 t} \quad (c_1 = 0, c_2 = 1)$$



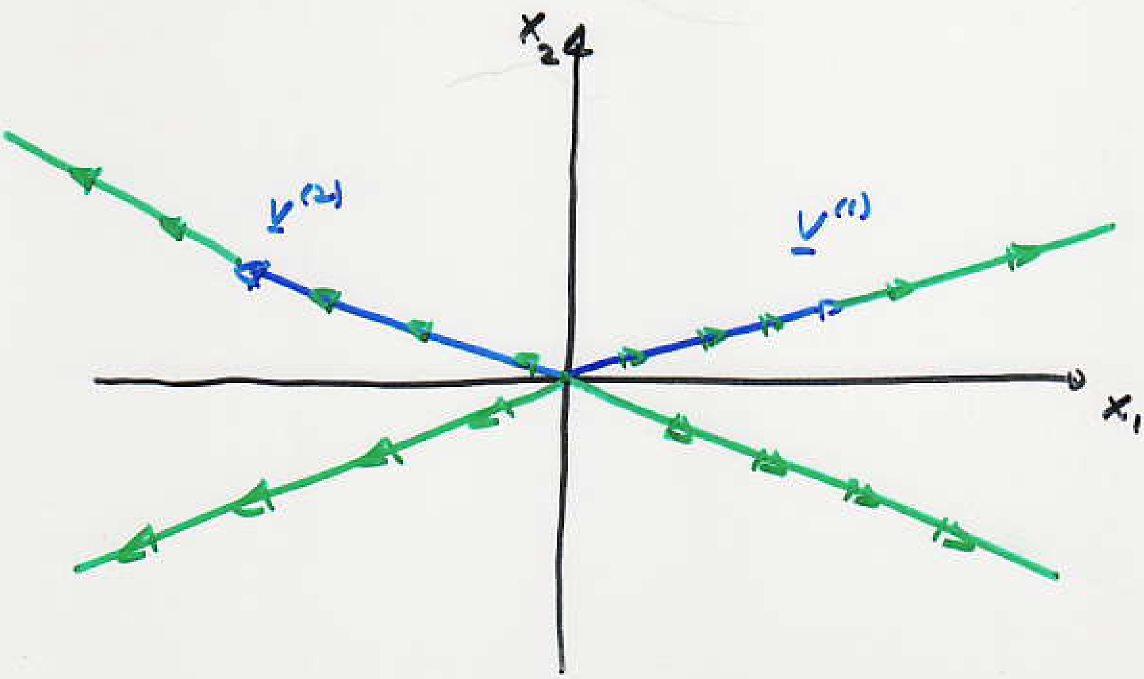
time is given by the arrows.

$\tilde{\underline{x}}(t) = -\underline{v}^{(1)} e^{\lambda_1 t}$, $\bar{\bar{\underline{x}}}(t) = -\underline{v}^{(2)} e^{\lambda_2 t}$

$(c_1 = -1,\ c_2 = 0)$ $(c_1 = 0,\ c_2 = -1)$

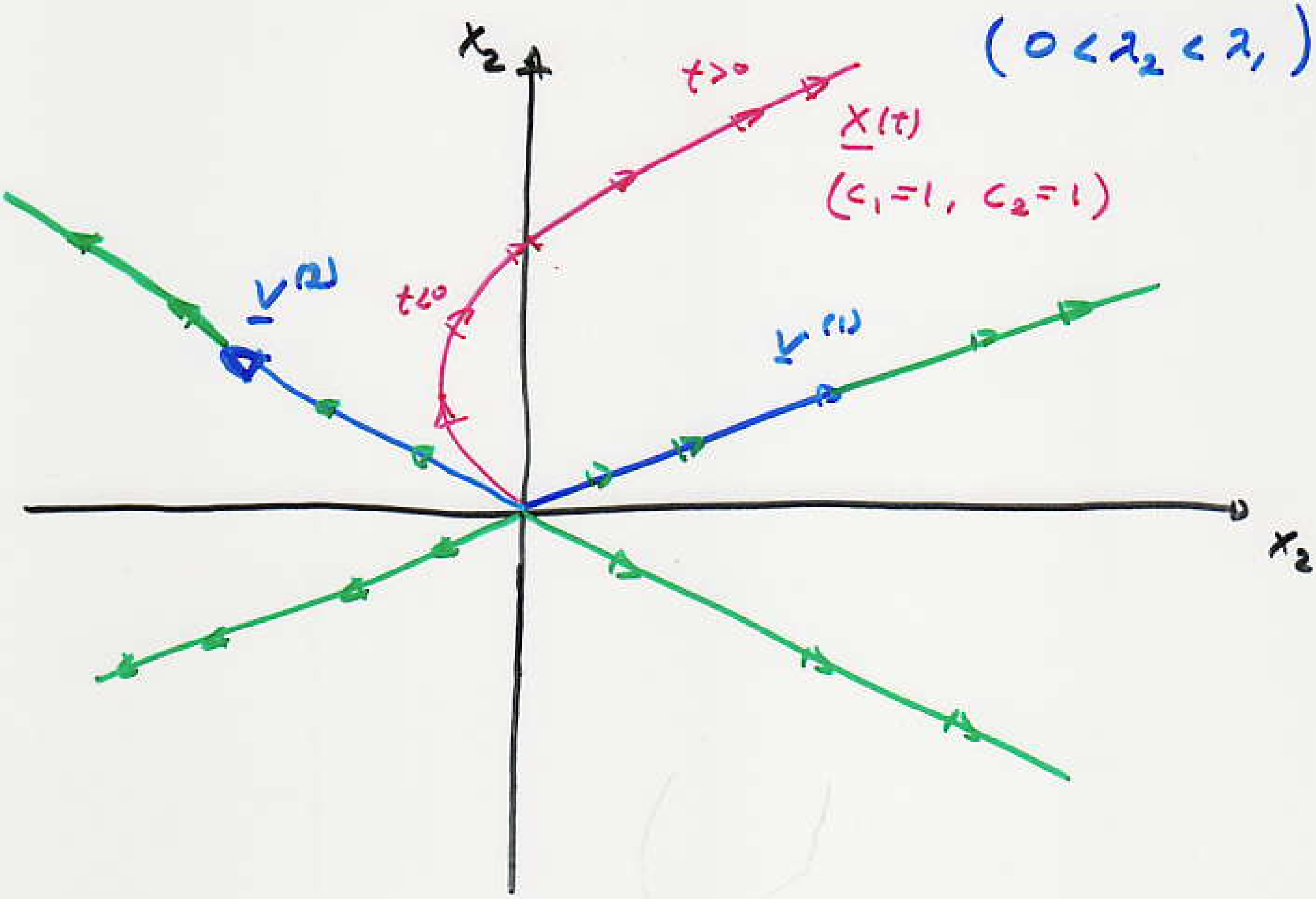


so, four different solutions are:

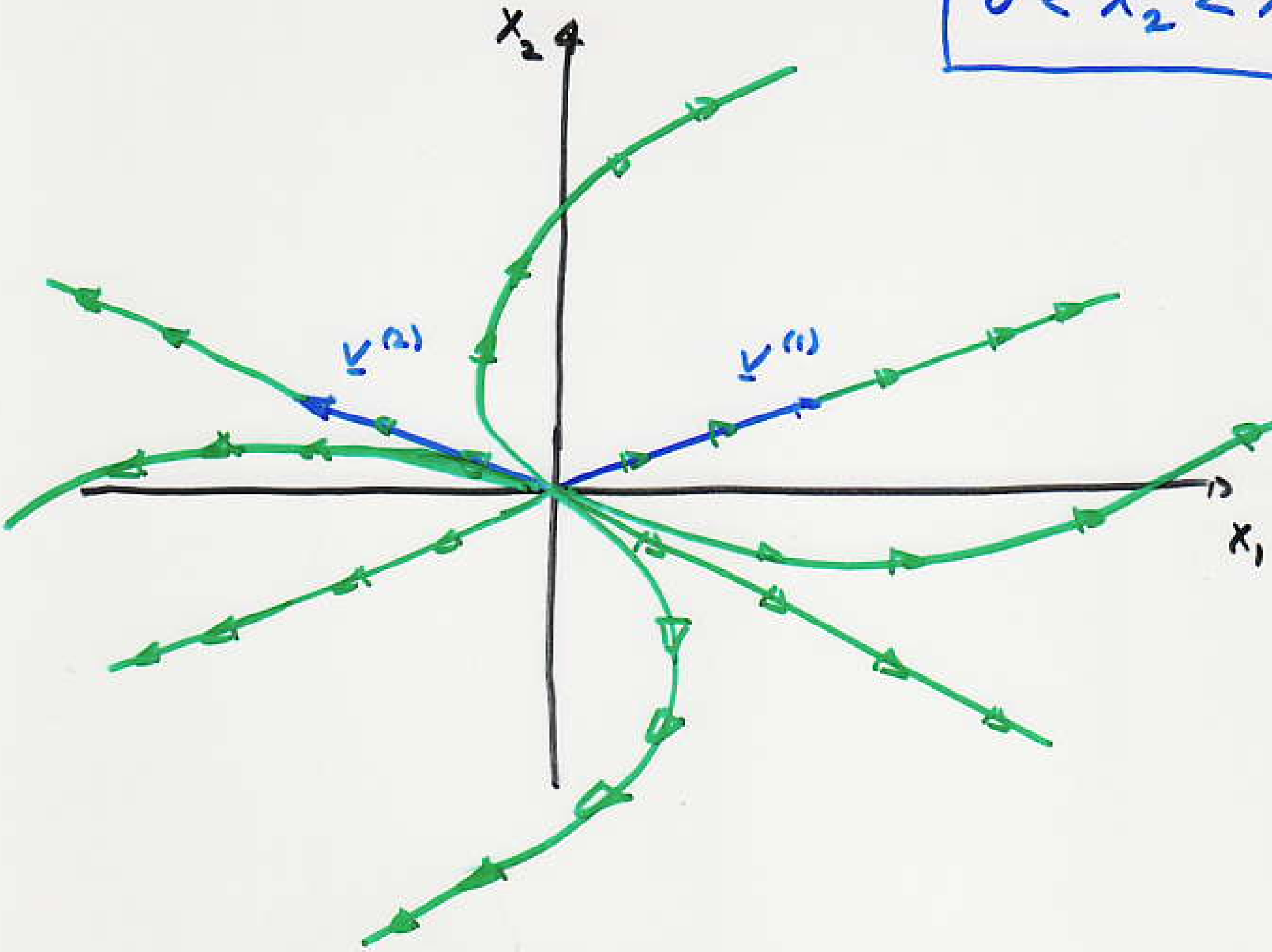


One more solution: $c_1 = 1$, $c_2 = 1$

$$\underline{x}(t) = \underline{v}^{(1)} e^{\lambda_1 t} + \underline{v}^{(2)} e^{\lambda_2 t}$$



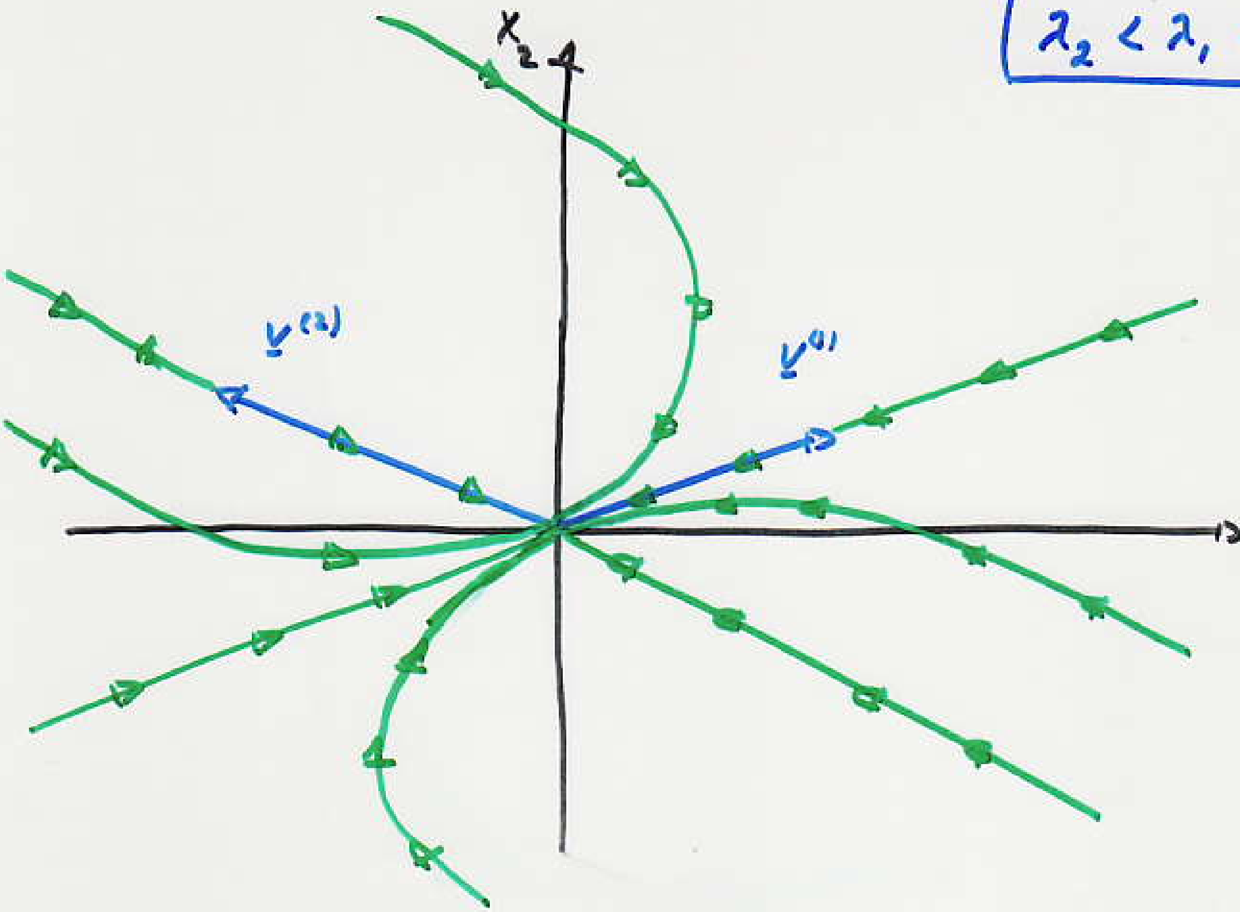
$$\boxed{0 < \lambda_2 < \lambda_1}$$



- eight solutions plotted.
- Arrows indicate time evolution (t)

$$\boxed{\underline{x} = \underline{0} \; : \; \text{unstable solution}}$$

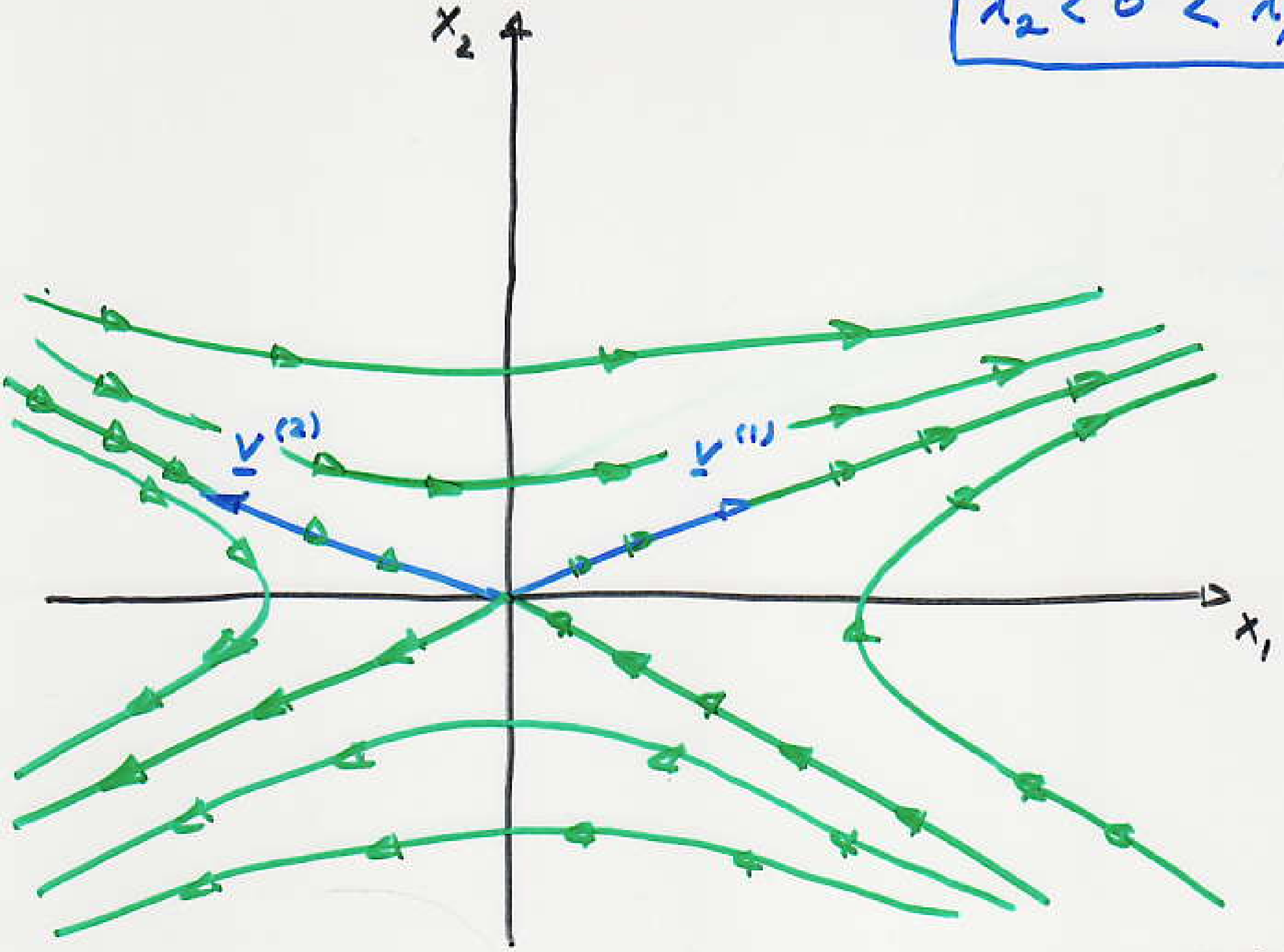
$\boxed{\lambda_2 < \lambda_1 < 0}$



- $\underline{x}(t) = c_1 \underline{v}^{(1)} e^{-|\lambda_1| t} + c_2 \underline{v}^{(2)} e^{-|\lambda_2| t}$
- eight solutions plotted.
- arrows indicate time evolution (t)

$\boxed{\underline{x} = \underline{0} \quad \text{stable solution}}$

$$\boxed{\lambda_2 < 0 < \lambda_1}$$



- $\underline{x}(t) = c_1 \underline{v}^{(1)} e^{|\lambda_1| t} + c_2 \underline{v}^{(2)} e^{-|\lambda_2| t}$
- ten solutions plotted
- arrows indicate time evolution (t)

$\boxed{\underline{x} = \underline{0} \quad \text{unstable solution}}$

(saddle point.)