mth 235 L 30

Plan: * General properties of $n \times n$ linear diff. eqs.

* The existence of fundamental sols. to homog. systems

* The Wronskian of sets of n solutions.

(7.4)

* Review: $n \times n$ linear diff. eqs.

Def: An $n \times n$ linear differential system is the following: Given an $n \times n$ matrix-valued function A and an n-vector-valued function $\underline{b}$, find an n-vector-valued function $\underline{x}$ sol. of

$$\boxed{\underline{x}'(t) = A(t)\,\underline{x}(t) + \underline{b}(t)} \qquad (1)$$

The system (1) is called homogeneous iff $\underline{b} = \underline{0}$

* **Example** : Express the differential syst. given by

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}$$

as a set of scalar eqs.

Sol :

Let $\quad \underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, then

$$\boxed{\underline{x}'(t) = A\,\underline{x}(t) + \underline{b}(t)}$$

means

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}$$

that is

$$\boxed{x_1'(t) = x_1(t) + 3x_2(t) + e^t}$$

$$\boxed{x_2'(t) = 3x_1(t) + x_2(t) + 2e^{3t}.}$$

* **Remark** :

- $$\underset{\sim}{x}'(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}' = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

- $$A'(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}'$$

$$= \begin{bmatrix} a_{11}'(t) & \cdots & a_{1n}'(t) \\ \vdots & & \vdots \\ a_{n1}'(t) & \cdots & a_{nn}'(t) \end{bmatrix}$$

Example : If $\underset{\sim}{x}(t) = \begin{bmatrix} e^{2t} \\ \sin(t) \\ \cos(t) \end{bmatrix}$,

then $\underset{\sim}{x}'(t) = \begin{bmatrix} 2e^{2t} \\ \cos(t) \\ -\sin(t) \end{bmatrix}$

Def : An $n \times n$ matrix-valued function with values $A(t) = [a_{ij}(t)]$ is called **continuous** iff every coefficient a_{ij} is a continuous function.

* The main result (without proof)

Thrm: (7.1.2) If the $n\times n$ matrix-valued function A and the n-vector-valued function $\underline{b}$ are continuous on $[t_0, t_1] \subset \mathbb{R}$, then for every n-vector $\underline{x}_0$ there exists a unique solution $\underline{x}(t)$ for $t \in [t_0, t_1]$ to the IVP

$$\boxed{\underline{x}'(t) = A(t)\,\underline{x}(t) + \underline{b}(t)} \qquad (1)$$

$$\boxed{\underline{x}(t_0) = \underline{x}_0 .} \qquad (2)$$

Remark :
- The eq (2) contains n initial conditions
- We will lear how to obtain such solutions
- We start studying homogeneous systems.

* Fundamental solutions to homogeneous diff. systs.

Def: A linearly independent (l. i.) set

$$\boxed{\{ \underline{x}^{(1)}(t), \ \ldots, \ \underline{x}^{(n)}(t) \}}$$

of solutions to an $n \times n$ homog. syst.

$$\boxed{\underline{x}'(t) = A(t)\, \underline{x}(t)} \qquad (3)$$

is called a fundamental set of solutions.

And the $n \times n$ matrix

$$\boxed{X(t) = [\underline{x}^{(1)}(t), \ \ldots, \ \underline{x}^{(n)}(t)]}$$

is called a fundamental matrix of system (3).

Def: The wronskian of a set of n n-vectors $\{\underline{x}^{(1)}(t), \ldots, \underline{x}^{(n)}(t)\}$ is given by

$$\boxed{W(t) = \det(X(t))}$$

where $X(t) = [\underline{x}^{(1)}(t), \ldots, \underline{x}^{(n)}(t)]$.

Propos. The solution set of n n-vectors $\{ \underline{x}^{(1)}(t), \ldots, \underline{x}^{(n)}(t) \}$ to an homog. diff. syst. is a fundamental set on the interval $I \subset \mathbb{R}$ iff

$$\boxed{W(t) = \det(X(t)) \neq 0}$$

for all $t \in I$.

Proof: From linear algebra we know that a set of vectors

$$\{ \underline{x}^{(1)}(t), \ldots \underline{x}^{(n)}(t) \} \quad \text{for fixed } t \in I,$$

is l.i. iff

$$\det\left(\left[\underline{x}^{(1)}(t), \ldots, \underline{x}^{(n)}(t) \right] \right) \neq 0$$

iff

$$\boxed{W(t) \neq 0.}$$

∎

Example: Show that the vector-valued functs.

$$\underline{x}^{(1)}(t) = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}, \quad \underline{x}^{(2)}(t) = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$$

form a l.i. set for all $t \in \mathbb{R}$.

Sol:

compute their Wronskian.

$$W(t) = \det(X(t)) = \det([\underline{x}^{(1)}(t), \underline{x}^{(2)}(t)])$$

$$= \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix}$$

$$= -2e^{-t}e^{3t} - 2e^{3t}e^{-t}$$

$$= -2e^{2t} - 2e^{2t}$$

$$\boxed{W(t) = -4e^{2t}}$$

$$\boxed{W(t) \neq 0 \quad \text{for all } t \in \mathbb{R}.}$$

<u>Example</u>: Show that the set

$$\left\{ \underline{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} \ , \ \underline{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} \right\}$$

is a fundamental set on $\mathbb{R}$ for the system

$$\underline{x}' = A\,\underline{x} \ , \qquad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

<u>Sol</u>:

(1) Verify that $x^{(1)}$ and $x^{(2)}$ are sols.

<u>For $x^{(1)}$</u>:

$$\boxed{x^{(1)'}(t) = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}}$$

Since $\underline{x}^{(1)}(t) = \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$, then

$$\underline{x}^{(1)'}(t) = \begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

And:

$$A\,x^{(1)}(t) = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

$$A\underline{x}^{(1)}(t) = \begin{bmatrix}6 & -2\\ 4 & -2\end{bmatrix} e^{2t}$$

$$= \begin{bmatrix}4\\ 2\end{bmatrix} e^{2t}$$

$$\boxed{A\underline{x}^{(1)}(t) = 2\begin{bmatrix}2\\ 1\end{bmatrix} e^{2t}}$$

Therefore: $\boxed{\underline{x}^{(1)\prime}(t) = A\underline{x}^{(1)}(t)}$

For $\underline{x}^{(2)}(t)$:

$$\boxed{\underline{x}^{(2)}(t) = -\begin{bmatrix}1\\ 2\end{bmatrix} e^{-t}}$$

$$A\underline{x}^{(2)}(t) = \begin{bmatrix}3 & -2\\ 2 & -2\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} e^{-t}$$

$$= \begin{bmatrix}3 & -4\\ 2 & -4\end{bmatrix} e^{-t}$$

$$= \begin{bmatrix}-1\\ -2\end{bmatrix} e^{-t}$$

$$\boxed{A x^{(2)}(t) = -\begin{bmatrix}1\\ 2\end{bmatrix} e^{-t}} \Rightarrow \boxed{\underline{x}^{(2)\prime} = A\underline{x}^{(2)}}$$

(2) Show that $\{ \underline{x}^{(1)}, \underline{x}^{(2)} \}$ is l.i.

compute the wronskian:

$$W(t) = \det(X(t)) = \det([\underline{x}^{(1)}, \underline{x}^{(2)}])$$

$$= \begin{vmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{vmatrix}$$

$$= 4 e^{2t} e^{-t} - e^{2t} e^{-t}$$

$$= 4e^{t} - e^{t}$$

$$\boxed{W(t) = 3e^{t}}$$

$$\boxed{W(t) \neq 0 \quad \text{for} \quad t \in \mathbb{R}.}$$

* A fundamental set determines all solutions

Propos. If $\{\underline{x}^{(1)}(t), \ldots, \underline{x}^{(n)}(t)\}$ is a fundamental set on the interval $[t_0, t_1] \subset \mathbb{R}$ to the homogeneous system

$$\boxed{\underline{x}'(t) = A(t)\,\underline{x}(t),} \qquad (3)$$

then any other solution $\underline{y}(t)$ to eq. (3) on the interval $[t_0, t_1]$ is given by

$$\boxed{\underline{y}(t) = c_1\,\underline{x}^{(1)}(t) + \cdots + c_n\,\underline{x}^{(n)}(t),}$$

for a unique set of constants $\{c_1, \ldots, c_n\}$.

Proof. (See textbook.)

Notation. If $\boxed{\underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}$ and

and $X(t) = [\underline{x}^{(1)}(t), \ldots, \underline{x}^{(n)}(t)]$

then

$$\boxed{\underline{y}(t) = X(t)\,\underline{c}}$$

* The previous result justifies the following definition:

Def: Given a fundamental set

$$\{ \underline{x}^{(1)}(t), \dots, \underline{x}^{(n)}(t) \}$$

of solutions to

$$\underline{x}'(t) = A(t)\, \underline{x}(t), \qquad (3)$$

the set of vectors of the form

$$\boxed{\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + \dots + c_n \underline{x}^{(n)}(t)}$$

for $c_1, \dots, c_n \in \mathbb{R}$, is called the general solution of (3).

Notation:

$$\boxed{\underline{x}(t) = X(t)\, \underline{c}}$$

$$= [\underline{x}^{(1)}(t), \dots, \underline{x}^{(n)}(t)] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\boxed{\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + \dots + c_n \underline{x}^{(n)}(t).}$$

<u>Example</u>: $\left[\begin{array}{l} \text{Find the sol. to the IVP} \\ x' = Ax, \quad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \\ \underline{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \end{array}\right]$

<u>Sol</u>:

- We need to find a fundamental set of solutions

- From the previous example we know:

$$\boxed{\left\{ \underline{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}, \quad \underline{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} \right\}}$$

is a fundamental set.

- So, the general sol. is

$$\underline{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

$$\boxed{\underline{x}(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$\boxed{\underline{x}(t) = X(t)\,\underline{c}}$$

- The initial condition is:

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \underline{x}(0) = X(0)\,\underline{c}$$

$$X(0)\,\underline{c} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad , \quad X(0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}}$$ solve for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(1) $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & 5 \end{array}\right] \longrightarrow$ find $\underline{c}$

(2) Find $[X(0)]^{-1} = \dfrac{1}{4-1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

So: $\underline{c} = \dfrac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$= \dfrac{1}{3} \begin{bmatrix} 2 - 5 \\ -1 + 10 \end{bmatrix}$

$= \dfrac{1}{3} \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

$$\boxed{\underline{c} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

$$\underline{x}(t) = X(t)\,\underline{c} = [\,\underline{x}^{(1)}(t), \underline{x}^{(2)}(t)\,]\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\boxed{\underline{x}(t) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}}$$

Remarks: - Next class we learn how to obtain fundamental solutions to homogeneous diff. systs. $\underline{x}' = A\,\underline{x}$ for A: constant.

- Eigenvalues and Eigenvectors play a fundamental role to find such solutions:

Notice: $\underline{x}^{(1)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$, $\underline{x}^{(2)}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

* A property of the Wronskian.

Propos. Given a continuous $n \times n$ matrix-valued function $A = [a_{ij}]$, the Wronskian of fundamental sols. to

$$\underline{x}'(t) = A(t)\,\underline{x}(t) \quad , \quad t \in [t_0, t_1]$$

is given by

$$\boxed{W(t) = W(t_0)\, e^{\int_{t_0}^{t} Tr(A)(\tilde{t})\, d\tilde{t}}}$$

where $Tr(A)(t) = a_{11}(t) + \dots + a_{nn}(t)$.

Notation: $Tr(A)$ is called the trace of A.

Remark: If $W(t_2) \neq 0$ at a single point $t_2 \in [t_0, t_1]$, then $W(t) \neq 0$ for all $t \in [t_0, t_1]$.

Proof for 2x2 systems

Denote: $\left\{ \underset{\sim}{x}^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{bmatrix}, \quad \underset{\sim}{x}^{(2)}(t) = \begin{bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{bmatrix} \right\}$

a fundamental set for $\underset{\sim}{x}' = A\underset{\sim}{x}$, 2x2, where

$$\boxed{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}.$$

Therefore $\underset{\sim}{x}^{(1)}$ satisfies:

$$\boxed{\begin{aligned} x_1^{(1)\prime} &= a_{11} x_1^{(1)} + a_{12} x_2^{(1)} \\ x_2^{(1)\prime} &= a_{21} x_1^{(1)} + a_{22} x_2^{(1)} \end{aligned}}$$

and $\underset{\sim}{x}^{(2)}$ satisfies

$$\boxed{\begin{aligned} x_1^{(2)\prime} &= a_{11} x_1^{(2)} + a_{12} x_2^{(2)} \\ x_2^{(2)\prime} &= a_{21} x_1^{(2)} + a_{22} x_2^{(2)} \end{aligned}}$$

The Wronskian is:

$$W(t) = \det\left(\left[\underset{\sim}{x}^{(1)}, \underset{\sim}{x}^{(2)} \right] \right)$$

$$= \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix}$$

$$W(t) = x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_1^{(2)}$$

so:

$$W'(t) = x_1^{(1)'} x_2^{(2)} + x_1^{(1)} x_2^{(2)'}$$
$$- x_2^{(1)'} x_1^{(2)} - x_2^{(1)} x_1^{(2)'}$$

$$= \left(a_{11} x_1^{(1)} + \underline{a_{12} x_2^{(1)}} \right) x_2^{(2)}$$
$$+ x_1^{(1)} \left(\underline{a_{21} x_1^{(2)}} + a_{22} x_2^{(2)} \right)$$
$$- \left(\underline{a_{21} x_1^{(1)}} + a_{22} x_2^{(1)} \right) x_1^{(2)}$$
$$- x_2^{(1)} \left(a_{11} x_1^{(2)} + \underline{a_{12} x_2^{(2)}} \right)$$

$$W' = a_{11} \left(x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_1^{(2)} \right) + a_{22} \left(x_1^{(1)} x_2^{(2)} - x_2^{(1)} x_1^{(2)} \right)$$

$$= a_{11} W + a_{22} W$$

$$\boxed{W'(t) = (a_{11} + a_{22}) W(t)}$$

$$\operatorname{Tr} A = a_{11} + a_{22}.$$

$$\boxed{W'(t) = (\operatorname{Tr} A) W(t)}$$

$$\boxed{W(t) = W(t_0) e^{\int_{t_0}^{t} \operatorname{Tr}(A)(\tilde{t})\, d\tilde{t}}}$$

∎