mth 235 L29

Plan:

* Eigenvalues - Eigenvectors of an $n \times n$ matrix.
* Computing Eigenvalues and Eigenvectors
* Algebraic and geometric multiplicities
* The case of Hermitian matrices.

(7.3)

* Eigenvalues and Eigenvectors

Def: A number λ and a non-zero n-vector $\underline{v}$ are called eigenvalue and eigenvector of an $n \times n$ matrix A iff holds

$$\boxed{A\underline{v} = \lambda \underline{v}}$$

Example: Show that $\lambda_1 = 4$, $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and $\lambda_2 = -2$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

are eigenvalues and eigenvectors of matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Sol:

$$\boxed{A\underline{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\,\underline{v}_1}$$

$$\boxed{A\underline{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = (-2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2\,\underline{v}_2}$$

* **Remark** :

- If we interpret $n \times n$ matrix A as a function $A: \mathbb{R}^n \to \mathbb{R}^n$, the eigenvector $\underline{v}$ determines a particular direction on $\mathbb{R}^n$, where the action of A is simple: [$A\underline{v}$ is proportional to $\underline{v}$]

- Matrices usually change the direction of vectors;

Example:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

point in different directions

- except for eigenvectors:

Example:

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>Example</u>: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{(Reflection along } x_1 = x_2 \text{ axis).}$$

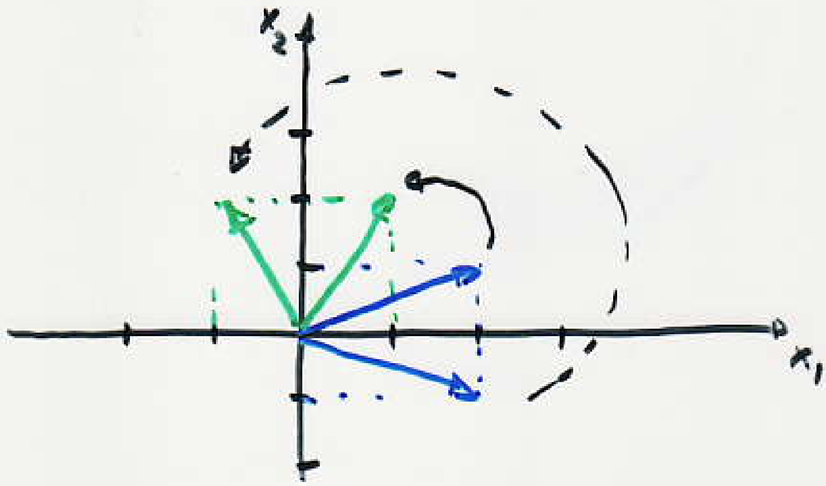
<u>Sol</u>:

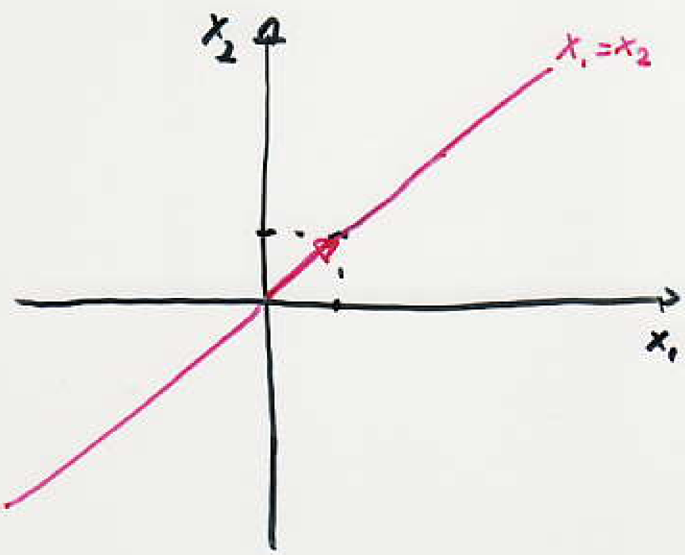
A is 2×2; $A : \mathbb{R}^2 \to \mathbb{R}^2$.

$$A\underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

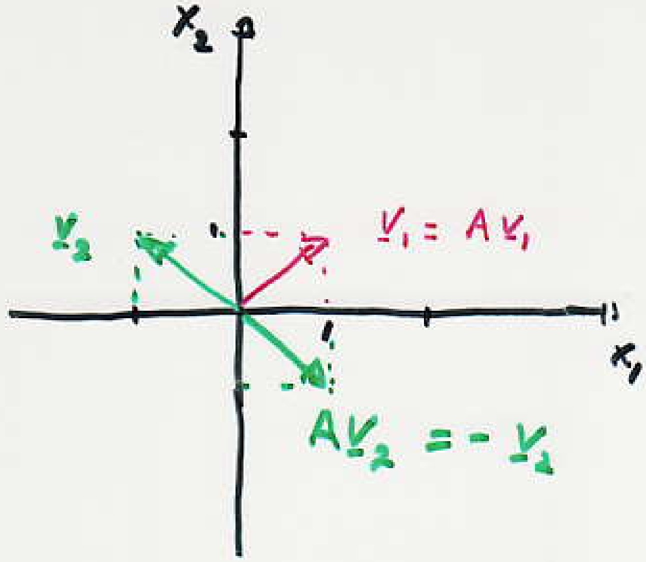
$$A \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$





$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

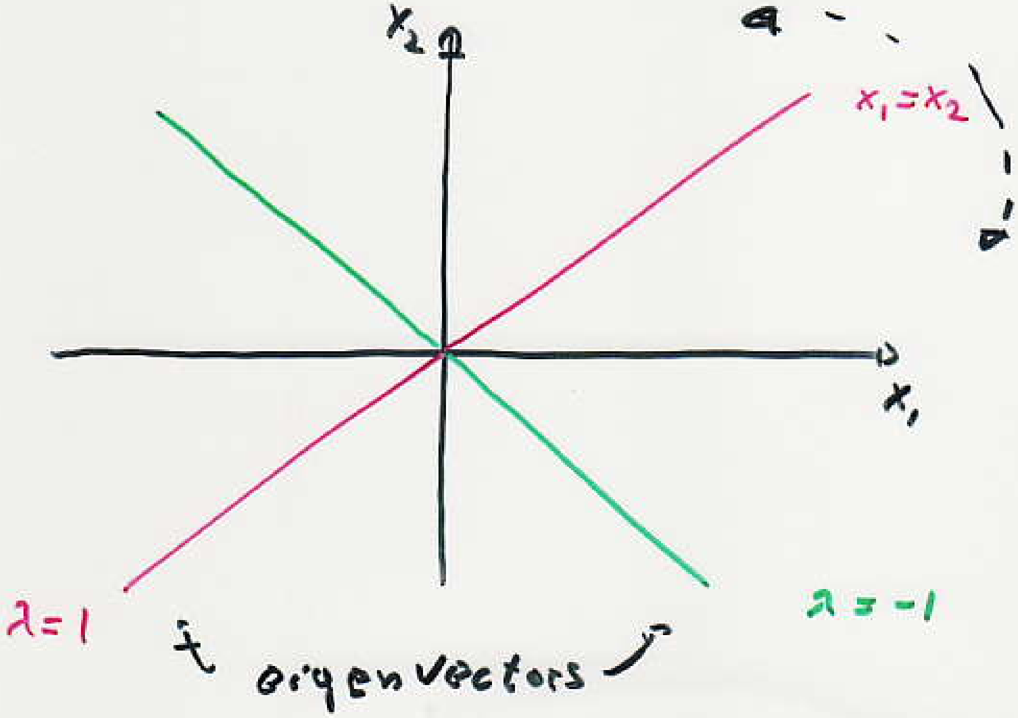
$$\boxed{\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_1 = 1}$$



$\lambda_1 = 1 \quad , \quad \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$



A : Reflection along $x_1 = x_2$

* **Remark**: Not every $n \times n$ matrix A has real eigenvalues:

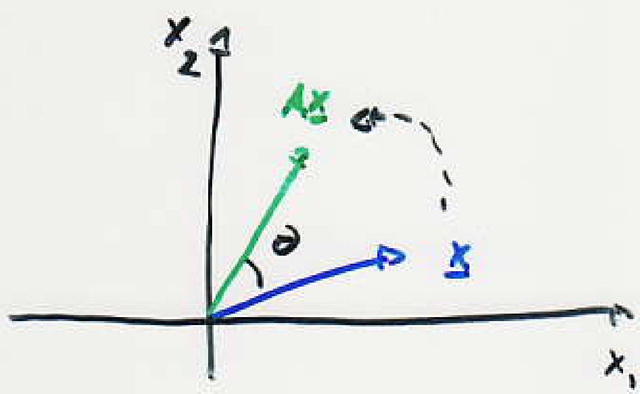
Example: Fix $\theta \in (0, \pi)$ and define

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Show that A has no real eigenvalues.

Sol:

$A : \mathbb{R}^2 \to \mathbb{R}^2$, A is a rotation by θ.



There is no direction kept fixed by A.

$\boxed{\text{So } A \text{ has no real eigenvalues - eigenvectors}}$

Remark: A has complex eigenvalues and eigenvectors

* Computing eigenvalues - eigenvectors

Problem: Given $n \times n$ A, find, if possible, λ and $\underline{v} \neq \underline{0}$, such that

$$\boxed{A \underline{v} = \lambda \underline{v}} \qquad (1)$$

This is more complicated than solving $A\underline{v} = \underline{b}$, since in our case (1) we do not know $\underline{v}$ nor the source $\underline{b} = \lambda \underline{v}$.

Sol: (a) First solve for λ.

(b) Having λ, then solve for $\underline{v}$.

Thrm (a): The number λ is an eigenvalue of $n \times n$ matrix A iff

$$\boxed{\det(A - \lambda I_n) = 0}$$

Notation: $P(\lambda) = \det(A - \lambda I_n)$

characteristic polynomial (degree n)

Remark: An eigenvalue λ is a root of the polynomial P.

Thrm (b): Given λ eigenvalue of A, the corresponding eigenvectors $\underline{v}$ are the non-zero sols. of

$$\boxed{(A - \lambda I)\,\underline{v} = \underline{0}}$$

Proof **(a)**: Find λ such that for a non-zero $\underline{v}$ holds

$$A\underline{v} = \lambda \underline{v}$$

$$A\underline{v} - \lambda \underline{v} = \underline{0}$$

$$\underbrace{(A - \lambda I_n)}_{\text{cannot be invertible.}} \underline{v} = 0$$

(If $(A-\lambda I_n)$ is invertible, then

$(A-\lambda I_n)^{-1}(A-\lambda I_n)\,\underline{v} = \underline{0} \iff \underline{v} = \underline{0}$.)

So: $(A-\lambda I_n)$ is NOT invertible.

$$\boxed{(A-\lambda I_n) \text{ Not invertible} \iff \det(A-\lambda I) = 0}$$

So λ must be sol. of

$$\boxed{\det(A-\lambda I_n) = 0}$$

$\blacksquare$

Example : $\left[\text{Find } \lambda, v \text{ for } A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \right]$

sol :

(a) $$A - \lambda I_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\boxed{A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix}}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9$$

$$\boxed{P(\lambda) = (\lambda - 1)^2 - 9}$$ characteristic polyn. (degree 2)

$$P(\lambda) = 0 \iff (\lambda - 1)^2 - 9 = 0$$

$$(\lambda_{\pm} - 1) = \pm 3$$

$$\lambda_{\pm} = 1 \pm 3 \Rightarrow \boxed{\lambda_1 = 4} \quad \boxed{\lambda_2 = -2.}$$

(b) $\lambda_1 = 4$

$$A - 4I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$

$(A - 4I)\,\underline{v} = \underline{0}$ $\leftarrow$ Gauss ops.

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} v_1 - v_2 = 0 \\ v_2 : \text{free.} \end{array}$$

choose $v_2 = 1 \Rightarrow v_1 = 1 \Rightarrow \boxed{\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\boxed{\lambda_1 = 1}$

(b) $\lambda_2 = -2.$

$$A + 2I_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$(A + 2I_2)\,\underline{v} = \underline{0}$ $\leftarrow$ Gauss ops.

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} v_1 + v_2 = 0 \\ v_2 : \text{free.} \end{array}$$

choose : $v_2 = 1 \Rightarrow v_1 = -1 \Rightarrow \boxed{\underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$\boxed{\lambda_2 = -2}$

Remarks : * Not every matrix $n\times n$ has eigenvectors

* Not every $n\times n$ matrix has a l.i. set of n vectors.

* Information useful to know the maximum number of eigenvectors forming a l.i. set : The algebraic and geometric multiplicities of an eigenvalue.

Def. Let $\lambda_1, \dots, \lambda_k$, $1 \le k \le n$ be the roots of the characteristic polynomial P of an $n\times n$ matrix, that is

$$P(\lambda) = (\lambda-\lambda_1)^{r_1} \cdots\cdots (\lambda-\lambda_k)^{r_k}.$$

The numbers $r_1, \dots, r_k$ are called the algebraic multiplicities of the eigenvalues $\lambda_1, \dots, \lambda_k$, respectively.

Example [Find the algebraic multiplicities of the eigenvalues of

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$]

Sol:

Recall:

$$P(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 9$$

Since $P(\lambda) = \lambda^2 - 2\lambda + 1 - 9$

$$= \lambda^2 - 2\lambda - 8$$

$$\boxed{P(\lambda) = (\lambda-4)(\lambda+2)}$$

$\lambda_1 = 4$, $\boxed{r_1 = 1}$

$\lambda_2 = -2$, $\boxed{r_2 = 1}$

Def: The *geometri multiplicity* of an eigenvalue λ_i, denoted by s_i, is the maximum number of eigenvectors with eigenvalue λ_i that form a l. i. set.

Example: Find the geometric multiplicities of the eigenvalues of

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Sol.

$$\lambda_1 = 4 \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2 \Rightarrow \boxed{s_1 = 1}$$

$$\lambda_2 = -2 \Rightarrow \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2 \Rightarrow \boxed{s_2 = 1}$$

Thrm: If λ is an eigenvalue of an $n \times n$ matrix A having algebraic multiplicity r and geometric multiplicity s, Then

$$\boxed{s \leq r}$$

<u>Example</u>: Find all eigenvalues, eigenvectors and multiplicities of

(a) $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Sol</u>:

case (a) - eigenvalues:

$$P(\lambda) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)^2 (1-\lambda)$$

$$P(\lambda) = -(\lambda-3)^2 (\lambda-1)$$

$\lambda_1 = 3$	$r_1 = 2$
$\lambda_2 = 1$	$r_2 = 1$

- eigenvectors: $\lambda_1 = 3$

$$(A - 3I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v_3 = 0$$

v_2, v_1 : free.

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_2 .$$

choose $v_1 = 1,\ v_2 = 0$ $\qquad \boxed{\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$ $\qquad \lambda_1 = 3.$

choose $v_1 = 0,\ v_2 = 1$ $\qquad \boxed{\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ l.i. eigenvectors set

for $\lambda_1 = 3.$ $\Rightarrow \boxed{s_1 = 2.}$

$(r_1 = s_1 = 2)$

$\lambda_2 = 1.$

$$A - I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} v_1 = -\frac{v_3}{2} \\ v_2 = -v_3 \\ v_3 : \text{free.} \end{array}$$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} v_3 \qquad \Rightarrow \boxed{s_2 = 1}$$

$(r_2 = s_2 = 1)$

choose $v_3 = 2$

$$\boxed{\underline{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \quad \lambda_2 = 1, \quad r_1 = 1, \quad s_1 = 1}$$

case (b) - eigenvalues: same as in (a).

$$P(\lambda) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)^2 (1-\lambda)$$

$$P(\lambda) = -(\lambda-3)^2 (\lambda-1) \Rightarrow \boxed{\begin{array}{ll} \lambda_1 = 3 & r_1 = 2 \\ \lambda_2 = 1 & r_2 = 1 \end{array}}$$

- eigenvectors:

$\lambda_1 = 3$

$$B - 3I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} v_2 = 0 \\ v_3 = 0 \\ v_1 : \text{free.} \end{array}$$

$$\boxed{\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_1} \Rightarrow \boxed{s_1 = 1}$$

$$\boxed{r_1 = 2} \qquad s_1 < r_1$$

choose $v_1 = 1$.

$$\boxed{\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_1 = 3, \quad r_1 = 2, \quad s_1 = 1}$$

$\lambda_2 = 1$

$$B - I_3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} v_1 = 0 \\ v_2 = -v_3 \\ v_3 : \text{free.} \end{array}$$

$$\underline{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} v_3 \quad \Rightarrow \quad \boxed{s_2 = 1}$$

$\boxed{r_2 = 1}$ $\quad s_2 = r_2.$

choose $v_3 = 1$

$$\boxed{\underline{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 1, \quad r_2 = 1, \quad s_2 = 1}$$

Eigenvectors of $\underset{3\times 3}{A}$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$ l.i.

Eigenvectors of $\underset{3\times 3}{B}$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ l.i.

* The case of Hermitian matrices

Def: [A, n×n is Hermitian iff

$$\boxed{A = A^*}$$]

Remark: [If A is Hermitian and real-valued, then

$$A = A^* = \bar{A}^T = A^T.$$]

Def: [A, n×n, (real- or complex-valued) is symmetric iff

$$\boxed{A = A^T}$$]

Remark: [If $A = [a_{ij}]$, and $A = A^T$,

Then $\boxed{a_{ij} = a_{ji}}$]

Example : Matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$ is symmetric,

since: $a_{12} = 3 = a_{21}$

$a_{13} = 2 = a_{31}$

$a_{23} = 4 = a_{32}$.

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$$

* Properties of Hermitian $n \times n$ matrices

(1) They always have real eigenvalues

(2) They always have a l.i. set of n eigenvectors.

(Spectral Decomposition Thrm.)