mth 235 L28

Plan: * Review: $n \times n$ algebraic linear systems

* Gauss elimination operations

* Computing the inverse of an $n \times n$ matrix

* Linear dependence - independence of vector sets.

* Eigenvalues - Eigenvectors of a matrix. (next class)

(7.3)

* **Review**: $n\times n$ algebraic systems

- Given an $n\times n$ matrix and n-vector

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$

find an n-vector $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, sol. of

$$\boxed{A\underline{x} = \underline{b}} \qquad (1)$$

- We are also interested in finding solutions to homogeneous systems

$$\boxed{A\underline{x} = \underline{0}}$$

- **Recall**: Augmented matrix of system (1):

$$[A \,|\, \underline{b}]$$

* **Example** : Given $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, find $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

sol. of

$$\boxed{A\underset{\sim}{x} = \underset{\sim}{b}}$$

That is:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

that is:

$$\begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

that is:

$$\boxed{\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned}}$$

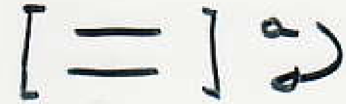
Augmented matrix : $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right]$.

Solution: Later.

* Gauss elimination operations

- These are operations on an n x n linear system that do not change its solutions but do change it augmented matrix.

(1) Add to one row a multiple of another row:



(2) Switch two rows

(3) Multiply a row by a non-zero number

* <u>Example</u>: [Use Gauss ops. to find x_1, x_2 sol. of

$$\boxed{\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned}} \qquad (1)$$

]

<u>Sol</u>:

Augmented matrix:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 2 & 3 \\ 2 & -1 & 0 \end{array}\right] \leftarrow (-1)$$

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 2 & -1 & 0 \end{array}\right] \begin{array}{c} (-2) \\ \end{array} \to \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 3 & 6 \end{array}\right] \leftarrow \frac{1}{3}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{c} \\ 2 \end{array} \to \boxed{\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]}$$

So system (1) has the same sols. as,

$$\boxed{\begin{aligned} x_1 + 0 &= 1 \\ 0 + x_2 &= 2. \end{aligned}} \iff \boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \end{aligned}}$$

<u>Remark</u>: Use gauss ops. to obtain a simple augmented matrix such that the sols. of the linear system are simple to obtain.

* **Example**: Show that the linear system below has no solutions,

$$\boxed{\begin{aligned} 2x_1 - x_2 &= 0 \\ -\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 &= -\tfrac{1}{4} \end{aligned}} \qquad (2)$$

Sol:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{array}\right] \leftarrow 4 \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & -1 \end{array}\right] (1)$$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right] \leftarrow (-1) \Rightarrow \boxed{\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]}$$

System (2) has the same sols. as

$$\boxed{\begin{aligned} 2x_1 - x_2 &= 0 \\ 0 &= 1 \end{aligned}} \Rightarrow \text{No solutions.}$$

Remark: If $[A \mid b] \longrightarrow [\tilde{A} \mid \tilde{b}]$ having a row $\boxed{[0 \cdots 0 \mid 1]}$, Then $A\underline{x} = \underline{b}$ has No sol.

* **Example**: Find all vectors $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $A\underline{x} = \underline{b}$ has solution $\underline{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \rightarrow$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & -1 & 1 & b_2+b_1 \\ 2 & -1 & 3 & b_3 \end{array}\right] \xrightarrow{(-1)} \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_2-b_1 \\ 2 & -1 & 3 & b_3 \end{array}\right] -2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_2-b_1 \\ 0 & 3 & -3 & b_3-2b_1 \end{array}\right] -3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_2-b_1 \\ 0 & 0 & 0 & b_3-2b_1+3b_1+3b_2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_2-b_1 \\ 0 & 0 & 0 & b_3+3b_2+b_1 \end{array}\right] \qquad (3)$$

$$b_3+3b_2+b_1 \neq 0 \iff (3) \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_2-b_1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

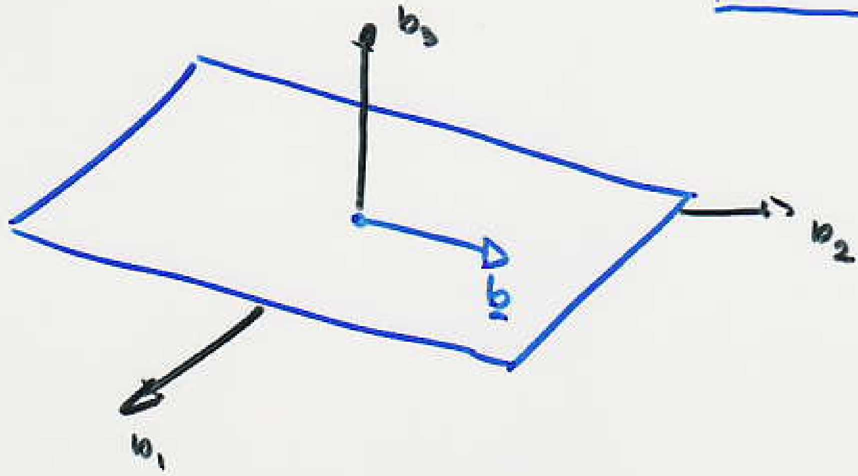
$\iff$ $A\underline{x} = \underline{b}$ No solution ($0=1$)

We conclude:

$$\boxed{A\underline{x} = \underline{b} \text{ has sol. } \underline{x} \iff b_3+3b_2+b_1 = 0}$$

<u>comment</u>: All possible source vectors $\underline{b}$ such that $A\underline{x} = \underline{b}$ has a solution $\underline{x}$ lie on a plane given by

$$\boxed{b_3+3b_2+b_1=0}$$



* **Example** : Find x_1 , x_2 , x_3 sol. of

$$x_1 + 2x_2 + x_3 = 1$$
$$-3x_1 + x_2 + 3x_3 = 24$$
$$x_2 - 4x_3 = -1$$

Sol.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -3 & 1 & 3 & 24 \\ 0 & 1 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 7 & 6 & 27 \\ 0 & 1 & -4 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & 7 & 6 & 27 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 34 & 34 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\boxed{\underline{x} = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}}$$

* **Computing the inverse of an $n \times n$ matrix.**

- Gauss operations can be used to find the inverse of an $n \times n$ matrix.

2x2 case: $A\,A^{-1} = I_2$

Denote: $A^{-1} = [\,\underline{x}_1, \underline{x}_2\,]$

then: $A\,[\underline{x}_1, \underline{x}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

that is:

$$A\,\underline{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\,\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Gauss ops. on: $\left[A \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix}\right], \left[A \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix}\right]$.

We can solve both systems at the same time with the augmented matrix

$$\left[A \,\middle|\, \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$$

That is: $\boxed{[A \mid I_2]}$

* **Example** : [use Gauss ops. to find the inverse of
$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$
]

Sol:

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -4 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3/4 & -1/2 \\ 0 & 1 & -1/4 & 1/2 \end{array}\right]$$

So:

$$A^{-1} = \begin{bmatrix} 3/4 & -1/2 \\ -1/4 & 1/2 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}}$$

* Example : Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$$

Sol:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}}$$

* <u>Application</u> <u>of</u> <u>determinants</u>

Thrm: [An $n \times n$ matrix A is invertible iff

$$\boxed{\det(A) \neq 0.}$$]

<u>Example</u>: [Is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$ invertible ?]

<u>Sol</u>:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{vmatrix} = (1)\begin{vmatrix} 5 & 7 \\ 7 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} + 3\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$$

$$= (45-49) - 2(18-21) + 3(14-15)$$

$$= -4 + 6 - 3$$

$$\boxed{\det(A) = -1} \quad \neq 0 \quad A \text{ is invertible.}$$

* Linearly dependent - independent vector sets.

Def: A set of n-vectors $\{\underline{v}_1, \dots, \underline{v}_k\}$, $k \geq 1$, is called *linearly dependent* (l.d.) iff there exist numbers $c_1, \dots, c_k$, with at least one of them non-zero, such that

$$\boxed{c_1 \underline{v}_1 + \cdots + c_k \underline{v}_k = \underline{0}}$$

Remarks:

- Suppose that the non-zero number is c_1. Then:

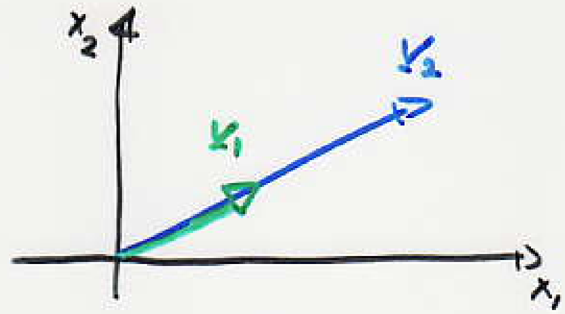
$$\boxed{\underline{v}_1 = -\frac{c_2}{c_1} \underline{v}_2 - \dots - \frac{c_k}{c_1} \underline{v}_k .}$$

- $\{\underline{v}_1, \dots, \underline{v}_k\}$ l.d., then one vector is a linear combination of the other vectors.

Examples :

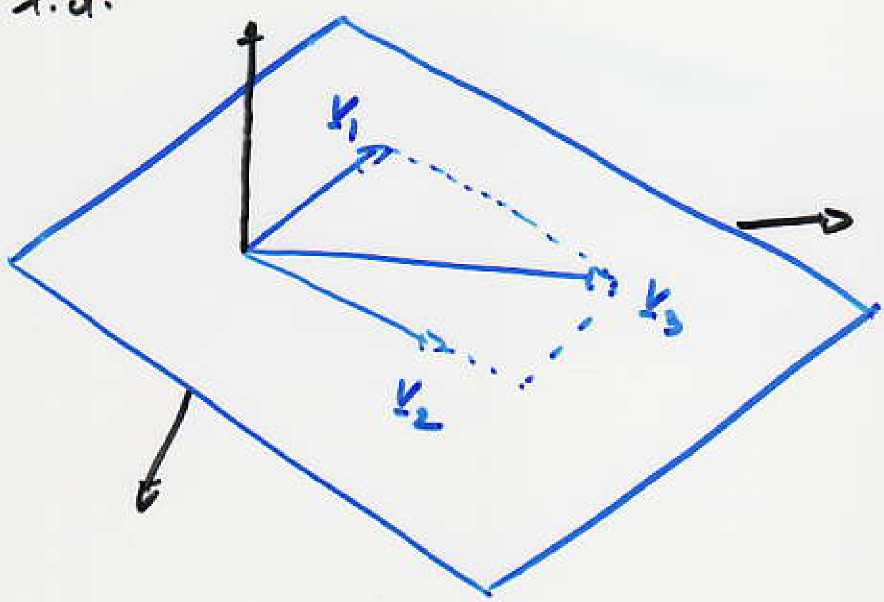
(1) A set of two co-linear vectors is l.d.

$$2\underline{v}_1 = \underline{v}_2$$



$$\boxed{2\underline{v}_1 - \underline{v}_2 = 0}$$

(2) A set of three co-planar vectors is l.d.



(3) The set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} \right\}$ is l.d.

Since $$2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Def: A set of n-vectors $\{\underline{v}_1, \dots, \underline{v}_k\}$, $k \geq 1$, is called *linearly independent*, (l.i.) iff the only linear combination of the form

$$\boxed{c_1 \underline{v}_1 + \cdots + c_k \underline{v}_k = \underline{0}}$$

is the one with $\boxed{c_1 = \cdots = c_k = 0.}$

Remark: A non-empty set of vectors is l.i $\iff$ it is not l.d.

Example: Show that $\left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is l.i.

Sol:

Find c_1, c_2 sol. of $\quad c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} c_1 - c_2 &= 0 \\ 2c_1 + c_2 &= 0 \\ 3c_1 &= 0 \quad \Rightarrow \boxed{c_1 = 0} \end{aligned}\right\} \Rightarrow \boxed{c_2 = 0}$$

$\boxed{\text{The set is l.i.}}$

Example Is the set $\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; v_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \; v_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ l.i. ?

Sol.

Find all c_1, c_2, c_3 sol. of:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} c_1 + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} c_2 + \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve the 3×3 system:

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} c_1 + 2c_3 &= 0 \\ c_2 - c_3 &= 0 \end{aligned} \Rightarrow \boxed{\begin{aligned} c_1 &= -2c_3 \\ c_2 &= c_3 \\ c_3 &: \text{free.} \end{aligned}}$$

We have found non-zero solutions c_1, c_2, c_3

Choose $\boxed{c_3 = 1} \Rightarrow \boxed{c_1 = -2}$, $\boxed{c_2 = 1}$

That means:

$$(-2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{The set is l.d.}}$

* Application of determinants

Thrm: The set of n-vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is l.i. iff

$$\boxed{\det[\underline{v}_1, \dots, \underline{v}_n] \neq 0.}$$

Example: Is $\left\{ \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ l.i ?

Sol.

$$\boxed{\begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 2 \\ 3 & 1 & 5 \end{vmatrix}} = (1)\begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= (10-2) - 3(10-6) - (2-6)$$

$$= 8 - 12 + 4$$

$$\boxed{= 0.}$$

$$\boxed{\text{The set is l.d.}}$$