mth 235 L 27

Plan: * The dot product of n-vectors

* The matrix-vector product

* A matrix is a function

* The inverse of a square matrix

* The determinant of a square matrix (7.2)

* nxn Systems of linear algebraic eqs. (7.3)

* **Review** : Matrix Multiplication:

- $\underset{m \times n}{A} \quad \underset{n \times l}{B} \longrightarrow \underset{m \times l}{AB}$

- Not commutative: In general $AB \neq BA$.

* The dot product.

- Matrix operations also apply on vectors, since an n-vector is an $n \times 1$ matrix.

Example: Given: $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 1+i \\ -1 \\ i \end{bmatrix}$:

(a) $\underline{u}^T = [1, 2, 3]$

$$\underline{u}^T \underline{u} = [1, 2, 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= 1^2 + 2^2 + 3^2$$

(b) $\underline{u}^T \underline{v} = [1, 2, 3] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$= -1 + 4 + 3$

$= 6$

(c) $\underline{w}^* \underline{w} = (\overline{\underline{w}})^T \underline{w}$

$= [1-i, -1, -i] \begin{bmatrix} 1+i \\ -1 \\ i \end{bmatrix}$

$= (1^2 + 1^2) + (-1)^2 + 1$

$= 4.$

Def: The **dot product** of two real or complex-valued n-vectors

$$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

is the number

$$\boxed{\underline{u} \cdot \underline{v} = \underline{u}^* \underline{v}}$$

Remark:

(a) $\underline{u} \cdot \underline{v} = (\bar{\underline{u}})^T \underline{v}$

$$\underline{u} \cdot \underline{v} = [\bar{u}_1, \dots, \bar{u}_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\boxed{\underline{u} \cdot \underline{v} = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n}$$

(b) For real-valued vectors

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = [u_1, \dots, u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\boxed{\underline{u} \cdot \underline{v} = u_1 v_1 + \cdots + u_n v_n.}$$

(c) $\underline{u}^* \underline{u} \geq 0$ for all $\underline{u} \in \mathbb{C}^n$, since

$$\underline{u}^* \underline{u} = \bar{u}_1 u_1 + \cdots + \bar{u}_n u_n$$

$$= |u_1|^2 + \cdots + |u_n|^2 \geq 0.$$

* The dot product provides a notion of vector length.

Def: The **norm** or length of an n-vector $\underline{u}$ is given by

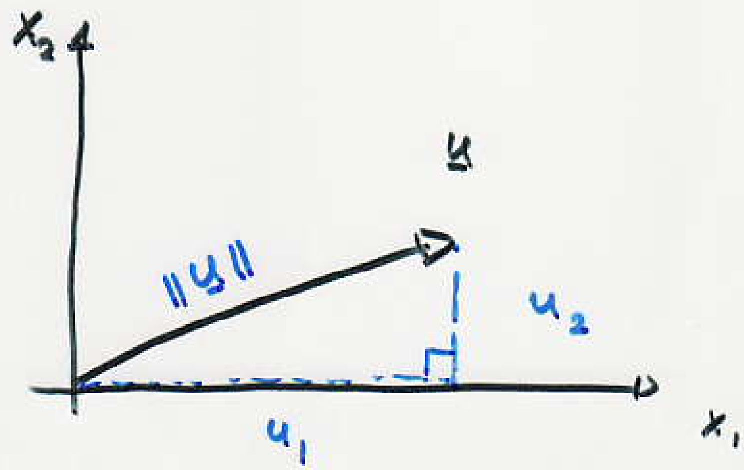
$$\boxed{\|\underline{u}\| = \sqrt{\underline{u}\cdot\underline{u}}}$$

Example: The norm of $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, real-valued,

is

$$\|\underline{u}\| = \sqrt{(u_1)^2 + (u_2)^2}$$

that is:



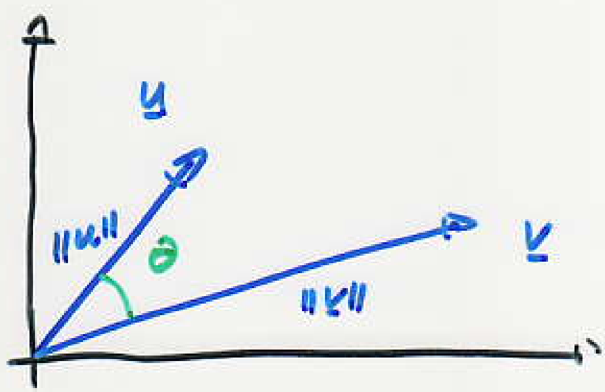
* The dot product provides a notion of perpendicular vectors

Thrm: $\boxed{\underline{u} \perp \underline{v} \iff \underline{u} \cdot \underline{v} = 0}$

The proof is based on the formula

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos(\theta)$$

where θ: angle between $\underline{u}$ and $\underline{v}$.



Example: $\left[\text{Find } \underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$

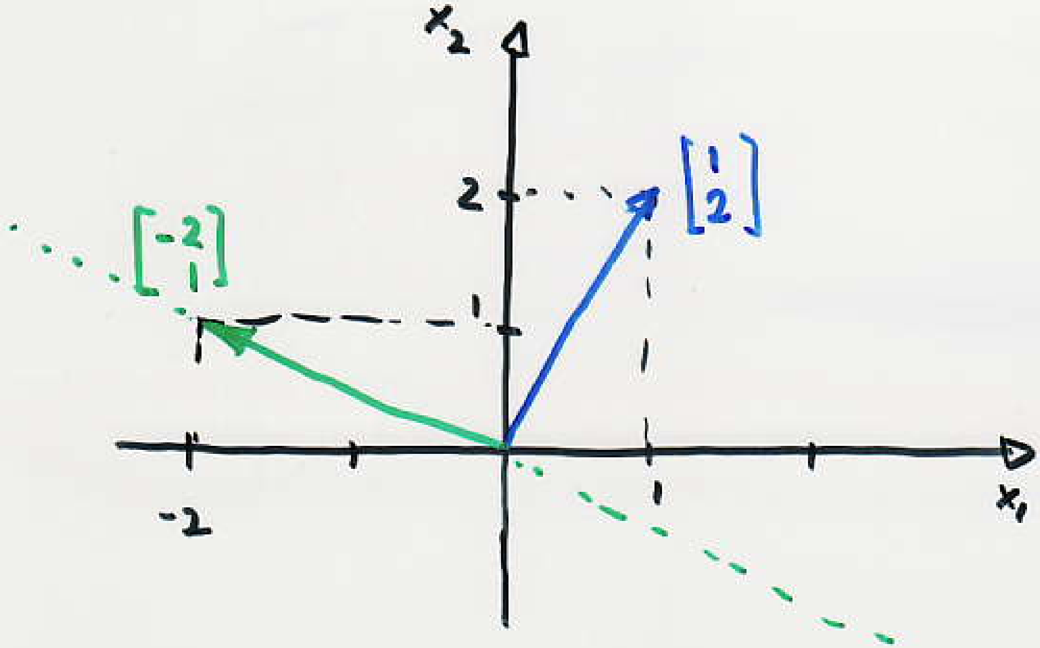
Sol:

$$[u_1, u_2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$$u_1 + 2u_2 = 0 \iff u_1 = -2u_2$$

So: $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -2u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} u_2$

$$\boxed{\underline{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} u_2}$$



* **The matrix-vector product**

Def: The matrix multiplication

$$\boxed{\underset{n\times n}{A} \quad \underset{n\times 1}{\underline{v}} \longrightarrow \underset{n\times 1}{A\underline{v}}}$$

is called the matrix-vector product of A and $\underline{v}$.

Example: Find $A\underline{v}$ for

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Sol:

$$A\underline{v} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-3 \\ -1+6 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$\boxed{A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}}$$

<u>Remark</u> :- The matrix-vector product provides a new interpretation for a matrix.

- An $n \times n$ matrix A is a function
$$A : \mathbb{R}^n \to \mathbb{R}^n$$
given by
$$\underline{v} \mapsto A\underline{v}$$

- A matrix is a function and matrix multiplication is equivalent to function composition.

Example: Show that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a rotation $\circlearrowleft$ by $90°$.

Sol:

A is $2\times 2 \Rightarrow A : \mathbb{R}^2 \to \mathbb{R}^2$.

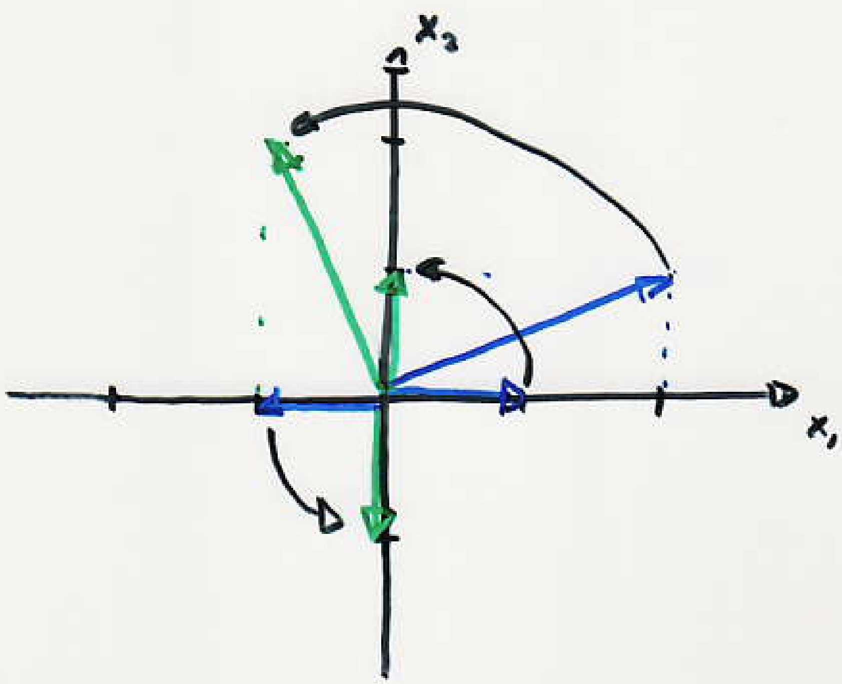
Given $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

$$A\underline{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$



Def : An $n \times n$ matrix I_n is called the identity matrix iff

$$\boxed{I_n \underline{x} = \underline{x}} \quad \text{for all } \underline{x} \in \mathbb{C}^n$$

Examples :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

* The inverse of a square matrix

Def: An $n \times n$ matrix A is called **invertible**, with inverse A^{-1}, iff there exists an $n \times n$ matrix A^{-1} satisfying

$$\boxed{A^{-1}A = I_n} \quad \text{and} \quad \boxed{AA^{-1} = I_n.}$$

Example: Show that the inverse of

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{is} \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

Sol:

$$AA^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6-2 & -2+2 \\ 3-3 & -2+6 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\boxed{AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \boxed{\text{Show } A^{-1}A = I_2.}$$

- Is every square matrix invertible?

NO.

Thrm: (2x2) Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $\Delta = ad - bc \neq 0$.

Furthermore, if A is invertible, then

$$\boxed{A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \qquad (1)$$

Proof (Furthermore only.)

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} da - bc & -ba + ba \\ dc - dc & -bc + ad \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$$

$$\boxed{AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Show $A^{-1}A = I_2$.

∎

Example: Find A^{-1} for $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Sol.

$$\Delta = 6 - 2 = 4$$

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \boxed{\frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}}.$$

Remark: The formula in (1) can be generalized to $n \times n$ matrices.

* The determinant of 2x2, 3x3 matrices.

Def: The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number

$$\boxed{\Delta = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

Remark: The determinant determines whether a matrix is invertible or not.

Examples

(1) $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$

(2) $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5$

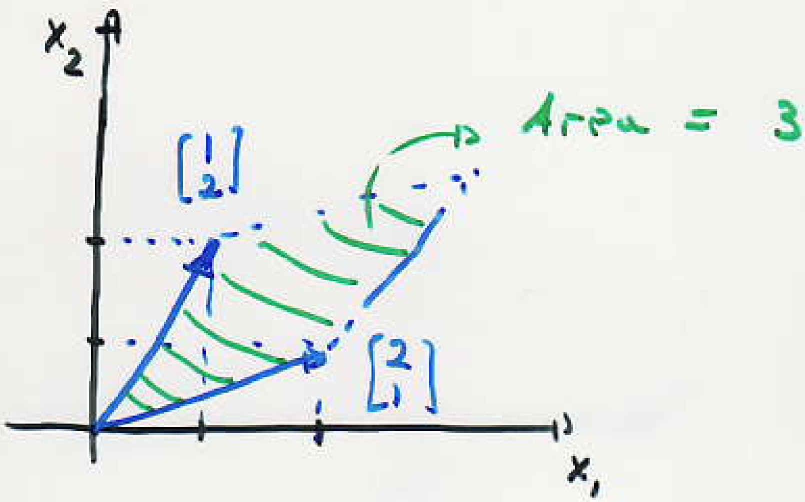
(3) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$

Remark :

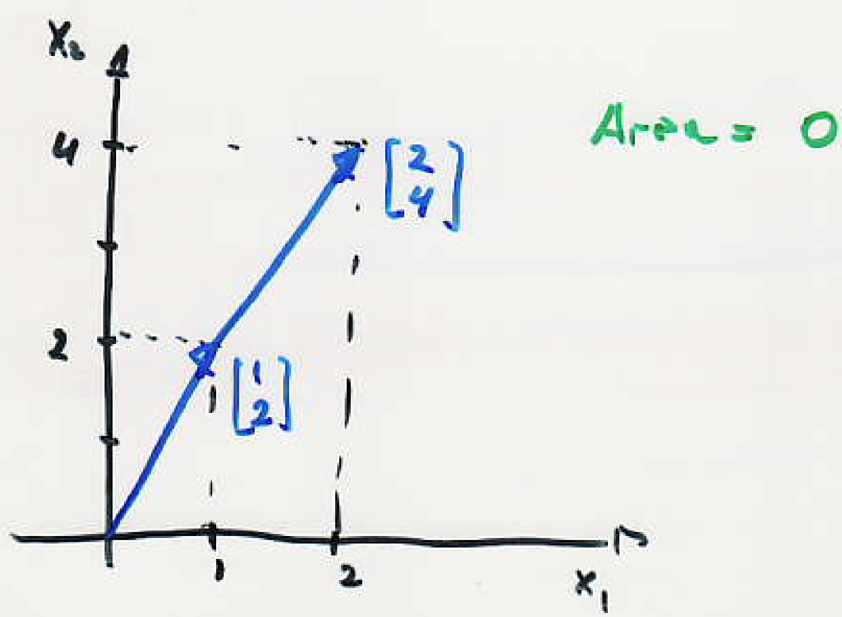
$$\left| \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right| = \text{Area of the parallelogram formed by } \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \begin{bmatrix} b \\ d \end{bmatrix}.$$

Example :

(1)



(3)



Def: The **determinant** of a 3x3 matrix A is given by:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example:

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (1-2) - 3(2-3) - (4-3)$$

$$= -1 + 3 - 1$$

$$\boxed{\det(A) = 1}$$

Remark: $|\det([\underline{x}_1, \underline{x}_2, \underline{x}_3])|$ = Volume of the Parallelopiped formed by $\underline{x}_1, \underline{x}_2, \underline{x}_3$.

* $n \times n$ Systems of Algebraic eqs.

Def: An $n \times n$ system of algebraic linear eqs. is the following: given numbers a_{ij}, b_i, $i, j = 1, \dots, n$, find numbers x_j sol. of

$$\boxed{\begin{aligned} a_{11} x_1 + \cdots + a_{1n} x_n &= b_1 \\ &\vdots \\ a_{n1} x_1 + \cdots + a_{nn} x_n &= b_n \end{aligned}} \qquad (1)$$

The system is called homogeneous iff

$$b_i = 0 \qquad i = 1, \dots, n.$$

Examples

(1) 2×2 :

$$\left[\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned}\right]$$

(2) 3×3

$$\left[\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -3x_1 + x_2 + 3x_3 &= 24 \\ x_2 - 4x_3 &= -1 \end{aligned}\right]$$

* Matrix notation is useful to work with systems of linear eqs. like (1)

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$ matrix of coefficients

$$\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$ source vector

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$ Unknown vector

Eq. (1) can be written as

$$\boxed{A\underline{x} = \underline{b}}$$

Augmented matrix : $[A \,|\, \underline{b}]$.

Example: The 2x2 system

$$\boxed{\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 &= 3 \end{aligned}} \qquad (2)$$

can be expressed as $\boxed{A\underline{x} = \underline{b}}$ with:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

that is,

$$\boxed{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}}$$

The augmented matrix of (2) is

$$\boxed{[A|\underline{b}] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right].}$$