mth 235 L26

Plan: * $n \times n$ systems of linear differential eqs.

* Reduction of a second order eq. into a 2×2 first order system (7.1)

* Main concepts from Linear Algebra (7.2)

* <u>Systems of linear differential eqs.</u>

- Many physical systems must be described with more than a single diff. eq.

- <u>Example</u>: Newton's law of motion:

$$\underline{x}''(t) = \frac{1}{m}\,\underline{F}(t, \underline{x}(t)) \quad ; \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Def: An $n \times n$ system of linear diff. eqs. is the following: Given the functions a_{ij}, $g_i : [a,b] \to \mathbb{R}$, with $i, j = 1, \dots, n$, find n functions $x_j : [a,b] \to \mathbb{R}$ solutions of n equations

$$\boxed{\begin{aligned} x_1' &= a_{11}(t)\, x_1 + \cdots + a_{1n}(t)\, x_n + g_1(t) \\ &\ \vdots \\ x_n' &= a_{n1}(t)\, x_1 + \cdots + a_{nn}(t)\, x_n + g_n(t) \end{aligned}}$$

The system is called homogeneous iff

$$g_i = 0, \qquad i = 1, \dots, n.$$

* Examples

(1) $n = 1$: single diff. eq. : Find x_1 sol. of

$$\left[\; x_1' = a_{11}(t)\, x_1 + g_1(t) \;\right]$$

(2) $n = 2$: 2×2 linear system: Find x_1, x_2, sol. of

$$\left[\; \begin{aligned} x_1' &= a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + g_2(t) \\ x_2' &= a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + g_2(t) \end{aligned} \;\right]$$

(3) $n = 2$: homogeneous $n \times n$ system: Find x_1, x_2 :

$$\left[\; \begin{aligned} x_1' &= a_{11}(t)\, x_1 + a_{12}(t)\, x_2 \\ x_2' &= a_{21}(t)\, x_1 + a_{22}(t)\, x_2 \end{aligned} \;\right]$$

$$(g_1 = g_2 = 0 .)$$

<u>Example</u>: Find x_1, x_2 sol. of 2x2 linear, homogeneous system

$$\begin{cases} x_1' = x_1 - x_2 \\ x_2' = -x_1 + x_2 \end{cases}$$

<u>Sol.</u>

Add up both eqs: $\boxed{(x_1 + x_2)' = 0}$

Subtract 2^{nd} from 1^{st}: $\boxed{(x_1 - x_2)' = 2(x_1 - x_2)}$

Denote: $v_1 = x_1 + x_2$, $\quad v_2 = x_1 - x_2$

then: $v_1' = 0 \Rightarrow \boxed{v_1 = c_1}$

$v_2' = 2v_2 \Rightarrow \boxed{v_2 = c_2 e^{2t}}$

Therefore:
$$x_1 = \frac{1}{2}(v_1 + v_2)$$
$$x_2 = \frac{1}{2}(v_1 - v_2)$$

So:
$$\boxed{x_1 = \frac{1}{2}(c_1 + c_2 e^{2t})}$$
$$\boxed{x_2 = \frac{1}{2}(c_1 - c_2 e^{2t})}$$

Transformation: 2nd order eq. into 1st order 2x2 system.

Lemma: A second order, linear, diff. eq. can be transformed into a 2x2 linear, first order, system.

Proof.

$$y'' + p(t)\, y' + q(t)\, y = g(t) \qquad (1)$$

Introduce:
$$\left.\begin{array}{l} x_1 = y \\ x_2 = y' \end{array}\right] \Rightarrow \boxed{x_1' = y' = x_2}$$

then, $x_2' + p(t)\, x_2 + q(t)\, x_1 = g(t)$.

we conclude:

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -q(t)\, x_1 - p(t)\, x_2 + g(t) \end{aligned}}$$

$\blacksquare$

<u>Example</u>: [Express the eq. below as a first order system.

$$y'' + 2y' + 2y = \sin(at)$$]

<u>Sol</u>:

Introduce:

$$\left.\begin{array}{l} x_1 = y \\ x_2 = y' \end{array}\right] \Rightarrow \quad x_1' = x_2$$

then:

$$x_2' + 2x_2 + 2x_1 = \sin(at).$$

so:

$$\boxed{\begin{array}{l} x_1' = x_2 \\ x_2' = -2x_1 - 2x_2 + \sin(at) \end{array}}$$

<u>Transformation</u>: 1^{st} order system into 2^{nd} order eq.

- [Systems of first orde equations can, sometimes, be transformed into a single eq.]

<u>Example</u>: [Express the 2×2 system below as a single diff. eq.

$$\boxed{\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= x_1 - x_2 . \end{aligned}}$$
]

<u>Sol</u>:

second eq.
$$\boxed{x_1 = x_2' + x_2}$$

then:
$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2$$

$$x_2'' + x_2' = -x_2' - x_2 + 3x_2$$

$$\boxed{x_2'' + 2x_2' - 2x_2 = 0 .}$$

- Find the solution x_1, x_2.

$r^2 + 2r - 2 = 0 \qquad r = \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm \sqrt{12}}{2}$

$$\boxed{r_\pm = -1 \pm \sqrt{3}}$$

$$\boxed{x_2 = c_1 e^{r_+ t} + c_2 e^{r_- t}}$$

$x_1 = x_2' + x_2$

$x_1 = c_1 r_+ e^{r_+ t} + c_2 r_- e^{r_- t} + c_1 e^{r_+ t} + c_2 e^{r_- t}$

$$\boxed{x_1 = c_1 (1 + r_+) e^{r_+ t} + c_2 (1 + r_-) e^{r_- t}}$$

* Review: Concepts from Linear Algebra.

- Ideas from linear algebra are useful to study systems of diff. eqs.

- We review:
  - $m \times n$ matrix
  - matrix operations
  - n-vectors, dot products.
  - matrix-vector product.

Def: An $m \times n$ matrix A is an array of numbers

$$A = \begin{array}{cc} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} & m \text{ rows} \\ n \text{ columns} & \end{array}$$

where the matrix coefficients $a_{ij} \in \mathbb{C}$, $\quad i = 1, \ldots, m$
$j = 1, \ldots, n$, $\quad m, n \geq 1$.

An $n \times n$ matrix is called a square matrix.

Examples:

(1) 2x2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(2) 2x3 matrix: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(3) 3x2 matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

(4) 2x2, complex matrix: $A = \begin{bmatrix} 1+i & 2-i \\ 3 & 4i \end{bmatrix}$

(5) The coefficients of a 2x2 linear sys.

$$\boxed{\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= x_1 - x_2 \end{aligned}}$$

can be grouped in the matrix

$$\boxed{A = \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}}$$

Def: An **m-vector** $\underline{v}$ is the array of numbers

$$\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$$

where the vector components $v_i \in \mathbb{C}$, $i = 1, \dots, m$.

<u>Comment</u>: An m-vector is an $m \times 1$ matrix

<u>Example</u>: The unknowns x_1, x_2 of the 2×2 linear diff. system

$$\boxed{\begin{aligned} x_1' &= -x_1 + 3x_2 \\ x_2' &= x_1 - x_2 \end{aligned}}$$

can be grouped in the 2-vector

$$\boxed{\underline{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}$$

* Operations with matrices

- We present only examples of the main matrix operations.

- consider the 2x3 matrix

$$\boxed{A = \begin{bmatrix} 1 & 2+i & -1+2i \\ 3i & 2 & 1 \end{bmatrix}} \qquad 2\times 3$$

(1)

$$A^T = \begin{bmatrix} 1 & 3i \\ 2+i & 2 \\ -1+2i & 1 \end{bmatrix} \qquad 3\times 2$$

A-transpose: interchange rows with colums.

- show: $(A^T)^T = A$.

(2) $$\bar{A} = \begin{bmatrix} 1 & 2-i & -1-2i \\ -3i & 2 & 1 \end{bmatrix} \qquad 2\times 3$$

$\boxed{A\text{-conjugate}}$: conjugate each matrix coefficient.

Recall : $\overline{a+bi} = a - bi$.

Remark : $\boxed{A \text{ Real iff } \bar{A} = A}$

$\boxed{A \text{ Imaginary iff } \bar{A} = -A}$

Show : $\overline{(\bar{A})} = A$.

(3) $A^* = (\bar{A})^T$ $\boxed{A\text{-adjoint.}}$

$$A^* = \begin{bmatrix} 1 & -3i \\ 2-i & 2 \\ -1-2i & 1 \end{bmatrix} \qquad 3\times 2.$$

* The addition and difference of two $m \times n$ matrices is defined componentwise.

- Given:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

(A, B size 2×3)

$$A + B = \begin{bmatrix} 1+1 & 3+2 & 5+3 \\ 2+3 & 4+1 & 6+2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 8 \\ 5 & 5 & 8 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-1 & 3-2 & 5-3 \\ 2-3 & 4-1 & 6-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 4 \end{bmatrix}$$

- Remark

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} : \text{ Not defined.}$$

* The multiplication of a matrix by a number is defined componentwise.

Given:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2(1) & 2(3) & 2(5) \\ 2(2) & 2(4) & 2(6) \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}$$

Also:

$$\begin{bmatrix} 9 & 27 \\ -18 & 81 \end{bmatrix} = 9 \begin{bmatrix} 1 & 3 \\ -2 & 9 \end{bmatrix}$$

And.

$$\frac{A}{3} = \begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{2}{3} & \frac{4}{3} & 2 \end{bmatrix}$$

* Matrix multiplication

Given $\underset{m \times n}{A} \quad \underset{n \times \ell}{B} \longrightarrow \underset{m \times \ell}{AB}$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow AB$$

$$2 \times 3 \qquad\qquad 3 \times 2 \qquad\qquad 2 \times 2$$

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1-2+3) & (2+2+6) \\ (3-2+1) & (6+2+2) \end{bmatrix}$$

$$\boxed{AB = \begin{bmatrix} 2 & 10 \\ 2 & 10 \end{bmatrix}} \qquad \left(= 2 \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix} \right)$$

Remark: $\boxed{BA \text{ is } 3 \times 3}$

So: $\boxed{AB \neq BA}$ Not commutative.

Remark: If AB and BA are both possible to compute, then in general holds

$$AB \neq BA.$$

Example: Find AB and BA for

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 6-2 & 0+1 \\ -3+4 & 0-2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6+0 & -3+0 \\ 4+1 & -2-2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}$$

$$AB \neq BA.$$

Example: [Find AB for

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$]

Sol:

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1+1 \\ -1+1 & 1-1 \end{bmatrix}$$

$$\boxed{AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \qquad A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Recall: $\boxed{a, b \in \mathbb{R} \text{ and } ab = 0 \Rightarrow a = 0 \text{ or } b = 0}$

$\uparrow$

Not true for matrices.