mth 235 L25

Plan:

* Convolution of two functions
* Laplace transform of a convolution
* Impulse response solution
* Solution decomposition theorem

(6.6)

* **Review:** The Dirac delta generalized function

$$\int_{-a}^{a} \delta(t)\, dt = 1$$

$$\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, dt = 1$$

$$\int_{c-a}^{c+a} \delta(t-c)\, f(t)\, dt = f(c)$$, $a>0$,
f : continuous

$$\mathcal{L}[\delta(t-c)] = e^{-cs}$$

$$\mathcal{L}[\delta(t)] = 1$$

* The convolution of two functions

Def: The convolution of $f, g : \mathbb{R} \to \mathbb{R}$ is a function $f * g : \mathbb{R} \to \mathbb{R}$ given by

$$\boxed{(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau}$$

Remark: - $f * g$ is also called a generalized product of f and g.

* Properties

Thrm:

$$
\begin{bmatrix}
(1) & f * g = g * f \\
(2) & f * (g+h) = f * g + f * h \\
(3) & f * (g * h) = (f * g) * h \\
(4) & f * 0 = 0 \\
(5) & f * \delta = f
\end{bmatrix}
$$

Proof (1):

$$\boxed{(f * g)(t)} = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$\tilde{\tau} = t - \tau \qquad d\tilde{\tau} = -d\tau$

$$= \int_t^0 f(t-\tilde{\tau})\, g(\tilde{\tau})\, (-d\tilde{\tau})$$

$$= \int_0^t g(\tilde{\tau})\, f(t-\tilde{\tau})\, d\tilde{\tau}$$

$$= \boxed{(g * f)(t)}$$

Proof (5)

$$\boxed{(f * \delta)(t)} = (\delta * f)(t)$$

$$= \int_0^t \delta(\tau)\, f(t-\tau)\, d\tau$$

$$= \boxed{f(t)}$$

* The Laplace transform of a convolution

Thm: If f, g have well-defined Laplace transforms $\mathcal{L}[f]$, $\mathcal{L}[g]$, then

$$\boxed{\mathcal{L}[f * g] = \mathcal{L}[f]\ \mathcal{L}[g]}$$

Proof

$$\mathcal{L}[f]\ \mathcal{L}[g] = \left[\int_0^\infty f(t)\, e^{-st}\, dt \right] \left[\int_0^\infty g(\tilde{t})\, e^{-s\tilde{t}}\, d\tilde{t} \right]$$

$$= \int_0^\infty g(\tilde{t})\, e^{-s\tilde{t}} \left[\int_0^\infty f(t)\, e^{-st}\, dt \right] d\tilde{t}$$

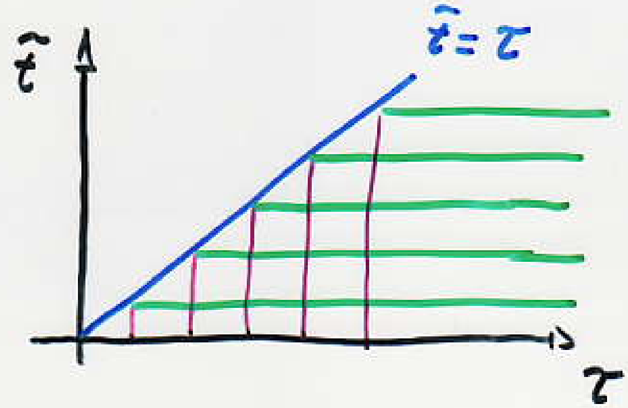
$$= \int_0^\infty g(\tilde{t}) \left[\int_0^\infty f(t)\, e^{-s(t+\tilde{t})}\, dt \right] d\tilde{t}$$

$$\tau = t + \tilde{t}$$

$$d\tau = dt$$

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde{t}) \left[\int_{\tilde{t}}^\infty f(\tau - \tilde{t})\, e^{-s\tau}\, d\tau \right] d\tilde{t}$$

$$= \int_0^\infty \int_{\tilde{t}}^\infty g(\tilde{t})\, f(\tau - \tilde{t})\, e^{-s\tau}\, d\tau\, d\tilde{t}$$



$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau g(\tilde{t})\, f(\tau - \tilde{t})\, e^{-s\tau}\, d\tilde{t}\, d\tau$$

$$= \int_0^\infty e^{-s\tau} \left[\int_0^\tau g(\tilde{t})\, f(\tau - \tilde{t})\, d\tilde{t} \right] d\tau$$

$$= \int_0^\infty e^{-s\tau}\, (f * g)(\tau)\, d\tau$$

$$\boxed{\mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[f * g]}$$

∎

* **Example** compute $\mathcal{L}[f]$, where

$$f(t) = \int_0^t e^{-(t-\tau)} \sin(\tau)\, d\tau.$$

Sol.

$$f(t) = \left(\frac{1}{\exp} * \sin\right)(t)$$

$$\mathcal{L}[f] = \mathcal{L}[e^{-t}]\ \mathcal{L}[\sin(t)]$$

$$= \frac{1}{(s+1)}\ \frac{1}{(s^2+1)}$$

$$\boxed{\mathcal{L}[f] = \frac{1}{(s+1)(s^2+1)}}$$

Example : Use convolutions to find $f(t)$ such that

$$\mathcal{L}[f] = \frac{s}{(s+1)\,(s^2+4)}$$

Sol.

$$\mathcal{L}[f] = \frac{1}{(s+1)}\;\frac{s}{(s^2+4)}$$

$$\mathcal{L}[f] = \mathcal{L}[e^{-t}]\;\mathcal{L}[\cos(2t)]$$

$$\boxed{f(t) = \int_0^t e^{-\tau}\cos(2(t-\tau))\,d\tau}$$

$$\boxed{f(t) = \int_0^t e^{-(t-\tau)}\cos(2\tau)\,d\tau}$$

$$f(t) = \frac{1}{\exp} * \cos(2\,\cdot) = \cos(2\,\cdot) * \frac{1}{\exp}$$

* **The impulse response solution**

Def: The function y_δ solution of the IVP

$$\boxed{\begin{array}{c|c} \multicolumn{2}{c}{y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t)} \\ \hline y_\delta(0) = 0 \;, & y_\delta'(0) = 0 \end{array}}$$

is called the impulse response solution.

Example Find the impulse response sol. of

$$y'' + 2y' + 2y = \delta(t)$$

$$y(0) = 0 \,, \quad y'(0) = 0$$

Sol:

$$\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\delta]$$

$$(s^2 + 2s + 2)\,\mathcal{L}[y] = 1$$

$$\boxed{\mathcal{L}[y] = \frac{1}{s^2 + 2s + 2}}$$

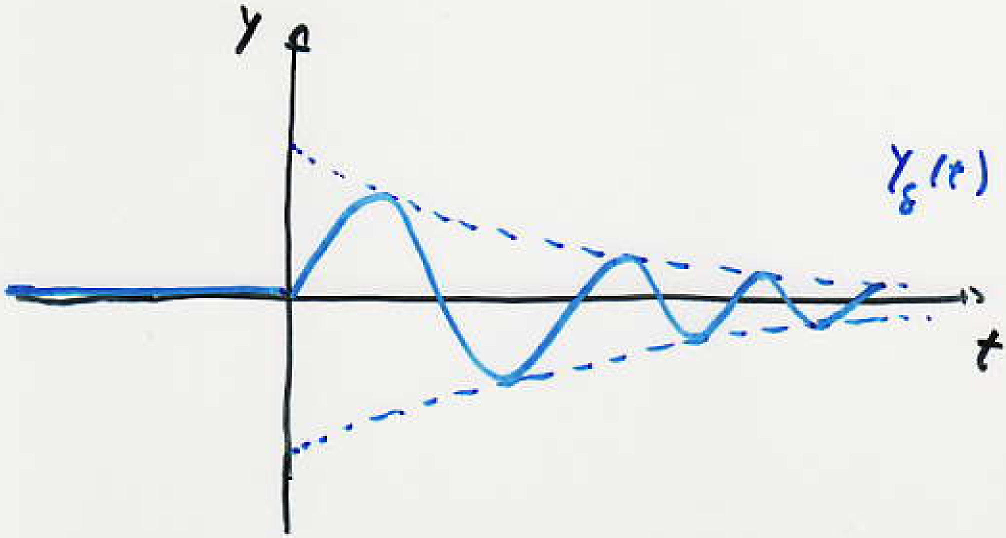
$s^2+2s+2=0 \quad \Rightarrow s = \frac{-2 \pm \sqrt{4-8}}{2}$ complex roots

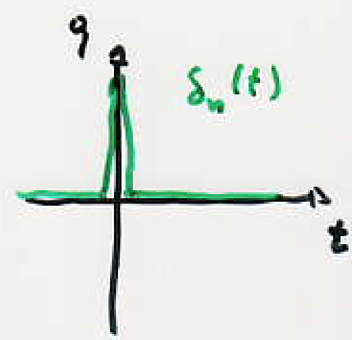
$$s^2+2s+2 = s^2+2s+1-1+2$$
$$= (s+1)^2+1$$

$$\boxed{\mathcal{L}[y] = \frac{1}{(s+1)^2+1}} \;;$$

$$\boxed{\mathcal{L}[\sin(t)] = \frac{1}{s^2+1}} \,, \quad \boxed{\mathcal{L}[f](s-c) = \mathcal{L}[e^{ct} f(t)]}$$

$$\boxed{y_\delta(t) = e^{-t}\sin(t)}$$





Example: Find the impulse response sol. to

$$y'' + 2y' + 2y = \delta(t-c)$$

$$y(0) = 0, \quad y'(0) = 0$$

Sol.

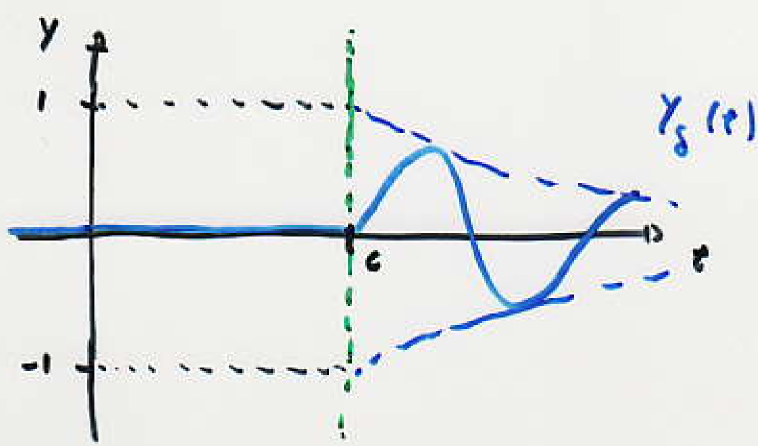
$$(s^2 + 2s + 2)\,\mathcal{L}[y] = \mathcal{L}[\delta(t-c)] = e^{-cs}$$

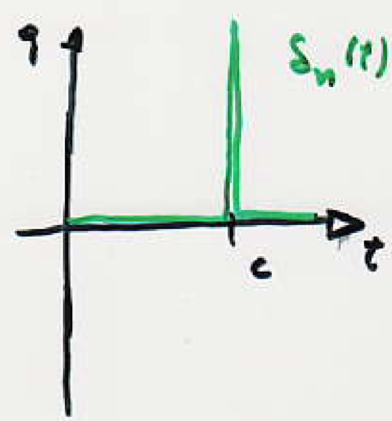
$$\boxed{\mathcal{L}[y] = \frac{e^{-cs}}{(s+1)^2 + 1}}$$

$$\mathcal{L}[e^{-t}\sin(t)] = \frac{1}{(s+1)^2+1},$$

$$e^{-cs}\,\mathcal{L}[f](s) = \mathcal{L}[u(t-c)\,f(t-c)]$$

$$\boxed{y_\delta(t) = u(t-c)\, e^{-(t-c)} \sin(t-c)}$$





* <u>The</u> <u>solution</u> <u>decomposition</u> <u>result.</u>

Thrm: The solution y to the IVP

$$\left[\begin{array}{l} y'' + a_1 y' + a_0 y = g(t) \\ y(0) = y_0 \quad , \quad y'(0) = y_1 \end{array}\right]$$

can be decomposed as

$$\boxed{y(t) = y_h(t) + (y_\delta * g)(t)}$$

where y_h is the solution of the homogeneous eq. and y_δ is the impulse response sol.; that is,

$$\left[\begin{array}{l} y_h'' + a_1 y_h' + a_0 y_h = 0 \\ y_h(0) = y_0 , \quad y_h'(0) = y_1 \end{array}\right]$$

and

$$\left[\begin{array}{l} y_\delta'' + a_1 y_\delta' + a_0 y_\delta = \delta(t) \\ y_\delta(0) = 0 \quad , \quad y_\delta'(0) = 0 . \end{array}\right]$$

Proof: **Recall:** $\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s\, y_0 - y_1$

$$\mathcal{L}[y'] = s\, \mathcal{L}[y] - y_0$$

Denote: $G(s) = \mathcal{L}[g(t)]$

$$\mathcal{L}[y''] + a_1 \mathcal{L}[y'] + a_0 \mathcal{L}[y] = \mathcal{L}[g]$$

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] - s\, y_0 - y_1 - a_1 y_0 = G(s)$$

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] = (s + a_1)\, y_0 + y_1 + G(s)$$

$$\boxed{\mathcal{L}[y] = \frac{(s + a_1)\, y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\, G(s)}$$

$$\boxed{\mathcal{L}[y_h] = \frac{(s + a_1)\, y_0 + y_1}{(s^2 + a_1 s + a_0)}}$$

$$\boxed{\mathcal{L}[y_g] = \frac{1}{(s^2 + a_1 s + a_0)}}$$

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\ \mathcal{L}[g]$$

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[(y_\delta * g)]$$

$$\boxed{y(t) = y_h(t) + (y_\delta * g)(t)}$$

■

$$\boxed{y(t) = y_h(t) + \int_0^t y_\delta(\tau)\, g(t-\tau)\, d\tau}$$

Example

$$\left[\begin{array}{l} \text{Find } y \text{ sol. of IVP} \\ y'' + 2y' + 2y = \sin(at) \\ y(0) = 1, \quad y'(0) = -1 \end{array}\right]$$

Sol:

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\sin(at)]$$

Recall:

$$\left.\begin{array}{l} \mathcal{L}[y''] = s^2 \mathcal{L}[y] - s(1) - (-1) \\ \mathcal{L}[y'] = s\,\mathcal{L}[y] - 1 \end{array}\right] \text{I.C.}$$

$$(s^2 + 2s + 2)\,\mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)]$$

$$\mathcal{L}[y] = \frac{(s+1)}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1}\,\mathcal{L}[\sin(at)]$$

$$\boxed{\mathcal{L}[y_h] = \frac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\cos(t)]}$$

$$\boxed{\mathcal{L}[y_\delta] = \frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\sin(t)]}$$

$$\mathcal{L}[y] = \underbrace{\mathcal{L}[e^{-t}\cos(t)]}_{\mathcal{L}[y_h]} + \underbrace{\mathcal{L}[e^{-t}\sin(t)]}_{\mathcal{L}[y_\delta]}\ \underbrace{\mathcal{L}[\sin(\omega t)]}_{\mathcal{L}[g]}$$

$$\boxed{y(t) = e^{-t}\cos(t) + \int_0^t e^{-\tau}\sin(\tau) \quad \sin\omega(t-\tau)\, d\tau}$$

$$\boxed{y(t) = y_h(t) + \int_0^t y_\delta(\tau) \quad g(t-\tau)\, d\tau}$$

$$\boxed{y(t) = y_h(t) + (y_\delta * g)(t)}$$