mth 235 L24

Plan:

* The Dirac delta generalized function

* Definition, properties

* Laplace transform of a Dirac's delta.

* Differential eqs. with Dirac's delta sources.

(6.5)

* Definition and properties of Dirac's delta

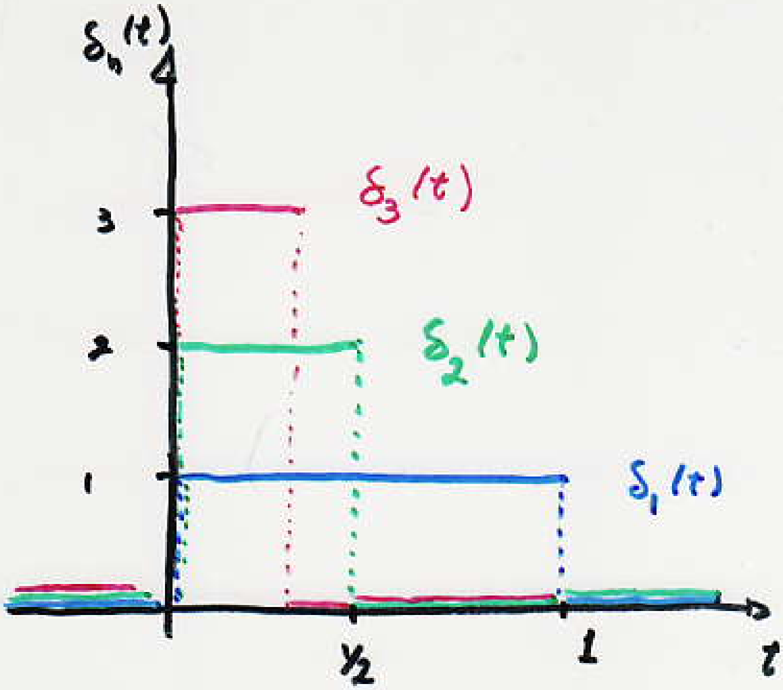
Def: Consider the sequence of functions

$$\delta_n(t) = \begin{cases} 0 & t < 0 \\ n & 0 \le t \le \frac{1}{n} \\ 0 & t > \frac{1}{n} \end{cases}.$$

The Dirac delta generalized function is given by

$$\boxed{\delta(t) = \lim_{n \to \infty} \delta_n(t).}$$

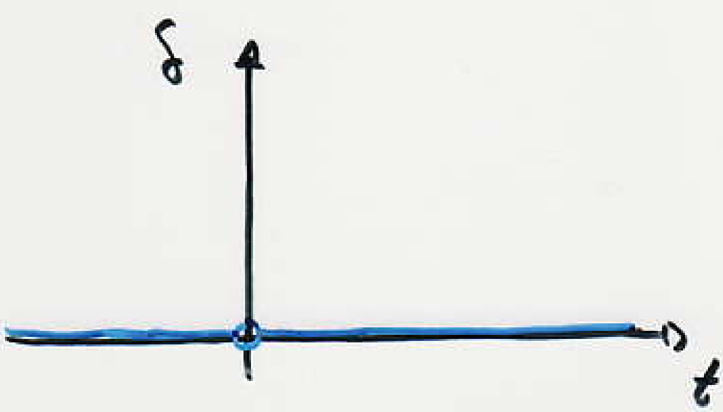
Remark:
- Different sequences δ_n define the same Dirac's δ.
- Compare with δ_n in textbook.



<u>Remark</u>: * δ is a function on the domain $\mathbb{R}-\{0\}$, and $\boxed{\delta(t)=0 \quad , \; t\in\mathbb{R}-\{0\}.}$

* $\boxed{\delta(0) = \lim_{n\to\infty} n = +\infty}$ <u>D.N.E.</u>

* $\boxed{\delta \text{ is not a function on } \mathbb{R}.}$



* Properties of Dirac's delta

- δ is not a function.

- Define: $$\boxed{\int_{-a}^{a} \delta(t)\, dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\, dt}$$

Thrm: $$\boxed{\int_{-a}^{a} \delta(t)\, dt = 1 \quad , \quad a > 0.}$$

Proof:
$$\int_{-a}^{a} \delta(t)\, dt = \lim_{n\to\infty} \int_{-a}^{a} \delta_n(t)\, dt$$
$$= \lim_{n\to\infty} \int_{0}^{1/n} n\, dt$$
$$= \lim_{n\to\infty} n \left(t\Big|_0^{1/n} \right)$$
$$= \lim_{n\to\infty} n \frac{1}{n}$$
$$\boxed{\int_{-a}^{a} \delta(t)\, dt = 1}$$

∎

Thrm: If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $t_0 \in \mathbb{R}$, and $a > 0$, then

$$\boxed{\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, f(t)\, dt = f(t_0).}$$

<u>Proof</u>:

$$\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, f(t)\, dt = \int_{-a}^{a} \delta(\tau)\, f(\tau+t_0)\, d\tau$$

$$\tau = t - t_0$$

$$= \lim_{n\to\infty} \int_{-a}^{a} \delta_n(\tau)\, f(\tau+t_0)\, d\tau$$

$$= \lim_{n\to\infty} \int_{0}^{1/n} n\, f(\tau+t_0)\, d\tau$$

Denote $f(t) = F'(t)$, $F(t) = \int f(t)\, dt$

$$\int_{t_0-a}^{t_0+a} \delta(t-t_0)\, f(t)\, dt = \lim_{n\to\infty} n \int_{0}^{1/n} F'(\tau+t_0)\, d\tau$$

$$= \lim_{n\to\infty} n \left(F(\tau+t_0) \Big|_{0}^{1/n} \right)$$

$$\int_{t_0-a}^{t_0+a} \delta(t-t_0) f(t)\, dt = \lim_{n\to\infty} n \left[F(t_0+\tfrac{1}{n}) - F(t_0) \right]$$

$$= \lim_{n\to\infty} \frac{F(t_0+\frac{1}{n}) - F(t_0)}{\frac{1}{n}}$$

$$= \lim_{h\to 0} \frac{F(t_0+h) - F(t_0)}{h}$$

$$= F'(t_0)$$

$$\boxed{\int_{t_0-a}^{t_0+a} \delta(t-t_0) f(t)\, dt = f(t_0)}$$

∎

* Relation between δ and u.

(Dirac's delta and step function.)

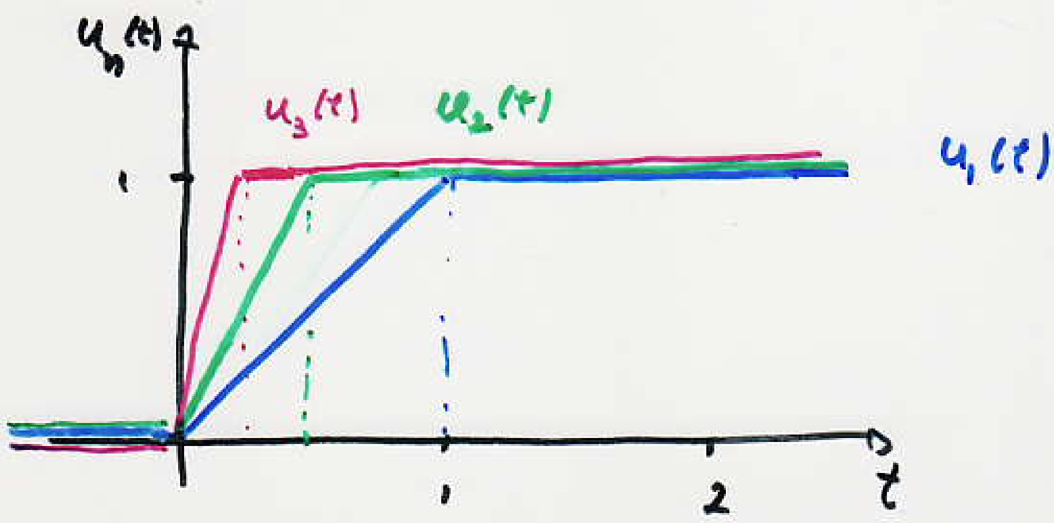
Proposition: The sequence of functions

$$u_n(t) = \begin{cases} 0 & t < 0 \\ nt & 0 \leq t \leq \frac{1}{n} \\ 1 & t > \frac{1}{n} \end{cases}$$

Satisfies both

$$\boxed{u_n'(t) = \delta_n(t)}$$

$$\boxed{\lim_{n \to \infty} u_n(t) = u(t)}$$



Remark: If we generalize the notion of derivative as

$$\boxed{\frac{d}{dt}u(t) = \lim_{n\to\infty} u_n'(t)}$$

then the proposition above says

$$\boxed{\delta(t) = \frac{d}{dt}u(t)}$$

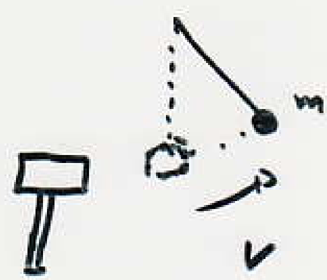
Remark: Dirac's delta is a generalized derivative of the step function.

* <u>comment</u>

- The Dirac delta generalized function is useful to describe impulsive forces in mechanical systems.

- An impulsive force transmits a finite momentum in an infinitely short time.

- Example: the momentum transmitted to a pendulum when we hit it with a hammer.



$$\boxed{m\,v' = F}$$

the action of the hammer can be described with a generalized function:

$$\boxed{F(t) = F_0 \;\delta(t-t_0)\,.}$$

$$\lim_{\Delta t \to 0} m\,v \Big|_{t_0}^{t_0+\Delta t} = \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} F(t)\,dt = F_0$$

* **The Laplace transform of δ**

Laplace transforms can be generalized from functions to δ.

We define:

$$\boxed{\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)]}$$

Thm:

$$\boxed{\mathcal{L}[\delta(t-c)] = e^{-cs}}$$

Proof:

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_0^\infty \delta_n(t-c)\, e^{-st}\, dt$$

$$= \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\, e^{-st}\, dt$$

$$= \lim_{n\to\infty} \left(-\frac{n}{s}\right) \left[e^{-s(c+\frac{1}{n})} - e^{-sc} \right]$$

$$\mathcal{L}[\delta(t-c)] = e^{-cs} \lim_{n\to\infty} \frac{\left(1 - e^{-s/n}\right)}{s/n} = \frac{0}{0}$$

$$= e^{-cs} \lim_{n\to\infty} \frac{-\frac{s}{n^2}\, e^{-s/n}}{\left(-\frac{s}{n^2}\right)}$$

$$= e^{-cs} \lim_{n\to\infty} e^{-s/n}$$

$$\boxed{\mathcal{L}[\delta(t-c)] = e^{-cs}}$$

∎

<u>Remark</u>: This result is consistent with the previous result:

$$\int_{t_0-\alpha}^{t_0+\alpha} \delta(t-t_0)\, f(t)\, dt = f(t_0)$$

$$\boxed{\mathcal{L}[\delta(t-c)] = \int_0^\infty \delta(t-c)\, e^{-st}\, dt = e^{-cs}}$$

$$\boxed{\mathcal{L}[\delta(t-c)\, f(t)] = \int_0^\infty \delta(t-c)\, e^{-st}\, f(t)\, dt = e^{-cs} f(c)}$$

* **Example** Find the sol. to IVP

$$y'' - y = -20\,\delta(t-3)$$

$$y(0) = 1, \quad y'(0) = 0$$

Sol.

$$\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)]$$

Recall

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)$$

so

$$s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) - \mathcal{L}[y] = -20\,e^{-3s}$$

$$(s^2-1)\,\mathcal{L}[y] - s\,y(0) - y'(0) = -20\,e^{-3s}$$

Initial cond.

$$(s^2-1)\,\mathcal{L}[y] - s = -20\,e^{-3s}$$

$$\boxed{\mathcal{L}[y] = \frac{s}{s^2-1} - 20\,e^{-3s}\,\frac{1}{s^2-1}}$$

Table: $\mathcal{L}[\cosh(at)] = \dfrac{s}{s^2 - a^2}$ $\quad s > |a|$

$\mathcal{L}[\sinh(at)] = \dfrac{a}{s^2 - a^2}$ $\quad s > |a|$

$$\boxed{\mathcal{L}[Y] = \mathcal{L}[\cosh(t)] - 20e^{-3s}\,\mathcal{L}[\sinh(t)]}$$

Recall: $e^{-cs} F(s) = \mathcal{L}[u(t-c)\, f(t-c)]$

So:

$$\mathcal{L}[Y] = \mathcal{L}[\cosh(t)] - 20\,\mathcal{L}[u(t-3)\sinh(t-3)]$$

$$\boxed{Y(t) = \cosh(t) - 20\,u(t-3)\sinh(t-3)}$$

Example : Find the sol. to IVP

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$$

$$y(0) = 0, \quad y'(0) = 0.$$

Sol.

$$\boxed{\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)]}$$

Recall

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)$$

Initial conds.

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y].$$

$$s^2\,\mathcal{L}[y] + 4\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s}$$

$$(s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s}$$

$$\boxed{\mathcal{L}[y] = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}}$$

Table: $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$

so:

$$\mathcal{L}[y] = \frac{e^{-\pi s}}{2}\,\frac{2}{s^2+2^2} - \frac{e^{-2\pi s}}{2}\,\frac{2}{s^2+2^2}$$

$$\boxed{\mathcal{L}[y] = \frac{1}{2}\left[e^{-\pi s}\,\mathcal{L}[\sin(2t)] - e^{-2\pi s}\,\mathcal{L}[\sin(2t)]\right]}$$

Recall: $e^{-\pi s}F(s) = \mathcal{L}[u(t-\pi)\,f(t-\pi)]$

so:

$$\mathcal{L}[y] = \frac{1}{2}\Big(\mathcal{L}[u(t-\pi)\sin(2(t-\pi))] - \mathcal{L}[u(t-2\pi)\sin(2(t-2\pi))]\Big)$$

$$y(t) = \frac{1}{2}\left[u(t-\pi)\sin(2t-2\pi) - u(t-2\pi)\sin(2t-4\pi)\right]$$

$$y(t) = \frac{1}{2}\left[u(t-\pi)\sin(2t) - u(t-2\pi)\sin(2t)\right]$$

$$\boxed{y(t) = \frac{1}{2}\left[u(t-\pi) - u(t-2\pi)\right] \sin(2t)}$$