mth 235 L23

Plan: * Solutions to differential eqs. with discontinuous sources

* Examples

(6.4)

* Differential eqs. with discontinuous sources

* Example : Find the sol. of IVP

$$y' + 2y = u(t-4)$$

$$y(0) = 0$$

Sol:

$$\boxed{\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[u(t-4)]}$$

Recall

$$\mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0) \quad , \quad y(0) = 0$$

$$\boxed{\mathcal{L}[y'] = s\,\mathcal{L}[y]}$$

$$s\,\mathcal{L}[y] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)]$$

$$(s+2)\,\mathcal{L}[y] = \frac{e^{-4s}}{s}$$

$$\boxed{\mathcal{L}[y] = \frac{e^{-4s}}{s(s+2)}}$$

Partial fractions

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{s+2}$$

$$= \frac{a(s+2) + bs}{s(s+2)}$$

$$\boxed{1 = (a+b)s + 2a} \Rightarrow \boxed{\begin{matrix} a+b=0 \\ 2a=1 \end{matrix}}$$

$$\boxed{a = \frac{1}{2}}, \quad \boxed{b = -\frac{1}{2}}$$

$$\boxed{\frac{1}{s(s+2)} = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right]}$$

$$y = \mathcal{L}^{-1}\left[\frac{e^{-4s}}{2}\left(\frac{1}{s} - \frac{1}{(s+2)}\right)\right]$$

$$\boxed{y = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s+2}\right]}$$

Recall: $\boxed{\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1}$ $\boxed{\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}}$

and

$$\boxed{\mathcal{L}^{-1}\left[e^{-cs} F(s)\right] = u(t-c)\, f(t-c)}$$

then

$$\boxed{\begin{aligned} \mathcal{L}^{-1}\left[\frac{e^{-4s}}{s}\right] &= u(t-4) \\ \mathcal{L}^{-1}\left[\frac{e^{-4s}}{s+2}\right] &= u(t-4)\, e^{-2(t-4)} \end{aligned}}$$

So:

$$y(t) = \frac{1}{2}\, u(t-4) - \frac{1}{2}\, u(t-4)\, e^{-2(t-4)}$$

$$\boxed{y(t) = \frac{1}{2}\, u(t-4) \left(1 - e^{-2(t-4)}\right)}$$

* **Example**: Find the sol. to IVP

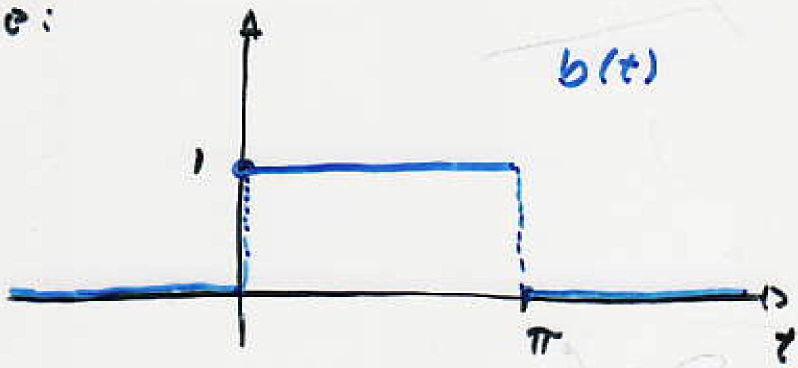
$$y'' + y' + \frac{5}{4} y = b(t)$$

$$y(0) = 0, \qquad y'(0) = 0$$

$$b(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$$

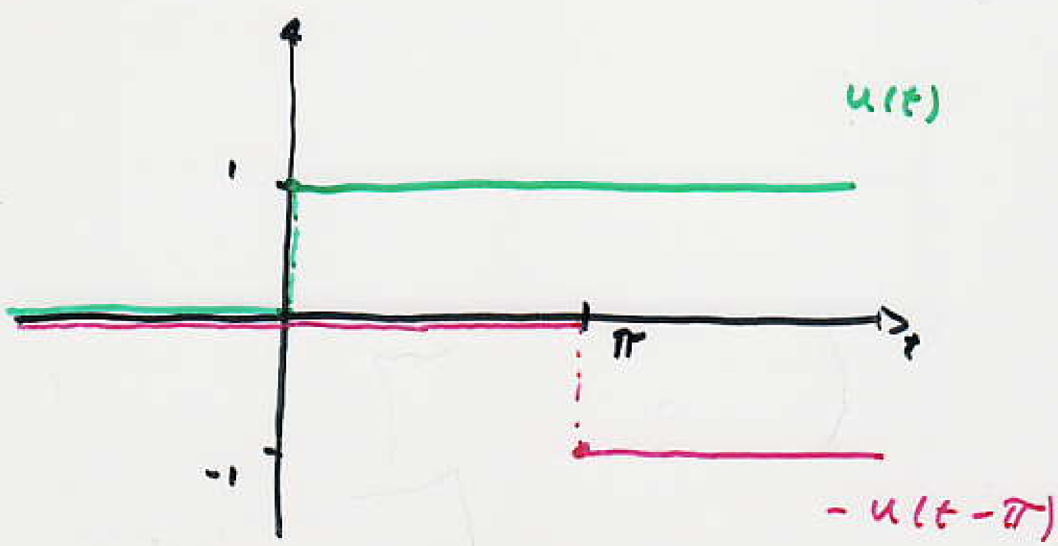
Sol.

The source:



Express b(t) using step functions

Recall:



$$\boxed{b(t) = u(t) - u(t-\pi)}$$

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\mathcal{L}[y] = \mathcal{L}[b(t)].$$

Recall

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y] - s\,y(0) - y'(0)$$

$$\mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0)$$

but: $y(0) = 0$, $y'(0) = 0$

$$\mathcal{L}[y''] = s^2 \mathcal{L}[y]$$

$$\mathcal{L}[y'] = s\,\mathcal{L}[y]$$

$$s^2 \mathcal{L}[y] + s\,\mathcal{L}[y] + \frac{5}{4}\mathcal{L}[y] = \mathcal{L}[b]$$

$$\boxed{\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \mathcal{L}[b(t)]}$$

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t-\pi)]$$

$$= \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$= \boxed{\frac{1}{s}(1 - e^{-\pi s})}$$

$$\left(s^2 + s + \frac{5}{4}\right)\mathcal{L}[y] = \frac{1}{s}\left(1 - e^{-\pi s}\right)$$

$$\boxed{\mathcal{L}[y] = \left(1 - e^{-\pi s}\right)\frac{1}{s\left(s^2 + s + \frac{5}{4}\right)}}$$

Rewrite denominator as a product of simpler polynomials.

(1) Find roots of denominator

$$s^2 + s + \frac{5}{4} = 0$$

$$s = \frac{-1 \pm \sqrt{1 - 5}}{2}.$$ No real roots.

(2) complete the square.

$$s^2 + s + \frac{5}{4} = s^2 + 2\,\frac{1}{2}s + \frac{5}{4}$$

$$= \underbrace{s^2 + 2\left(\frac{1}{2}\right)s + \frac{1}{4}} - \frac{1}{4} + \frac{5}{4}$$

$$= \boxed{\left(s + \frac{1}{2}\right)^2 + 1}$$

$$\mathcal{L}[y] = \left(1 - e^{-\pi s}\right) \frac{1}{s\left[\left(s+\frac{1}{2}\right)^2 + 1\right]}$$

$$\boxed{y = \mathcal{L}^{-1}\left[\frac{1}{s\left[\left(s+\frac{1}{2}\right)^2 + 1\right]}\right] - \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s\left[\left(s+\frac{1}{2}\right)^2 + 1\right]}\right]}$$

Denote: $\boxed{H(s) = \dfrac{1}{s\left[\left(s+\frac{1}{2}\right)^2 + 1\right]}}$

$$y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}\left[e^{-\pi s} H(s)\right]$$

Denote $\quad h(t) = \mathcal{L}^{-1}[H(s)]$.

then $\quad \mathcal{L}^{-1}[e^{-\pi s} H(s)] = u(t-\pi)\, h(t-\pi)$

So:

$$\boxed{y(t) = h(t) - u(t-\pi)\, h(t-\pi)}$$

[We only need to find $h(t)$.]

$$\boxed{h(t) = \mathcal{L}^{-1}\left[\frac{1}{s\left[(s+\frac{1}{2})^2+1\right]} \right]}$$

The computation of $h(t)$.

- Partial fractions

$$\frac{1}{s[(s+\frac{1}{2})^2+1]} = \frac{a}{s} + \frac{bs+c}{(s+\frac{1}{2})^2+1}$$

$$= \frac{a[(s+\frac{1}{2})^2+1] + s(bs+c)}{s[(s+\frac{1}{2})^2+1]}$$

$$1 = a(s^2+s+\tfrac{5}{4}) + bs^2 + cs$$

$$\boxed{1 = (a+b)s^2 + (a+c)s + \tfrac{5}{4}a}$$

$$\boxed{\begin{array}{l} a+b=0 \\ a+c=0 \\ \frac{5}{4}a=1 \end{array}} \qquad \boxed{a = \frac{4}{5}, \quad b = -\frac{4}{5}, \quad c = -\frac{4}{5}}$$

$$\boxed{\frac{1}{s[(s+\frac{1}{2})^2+1]} = \frac{4}{5}\left[\frac{1}{s} - \frac{(s+1)}{[(s+\frac{1}{2})^2+1]}\right]}$$

$$h(t) = \frac{4}{5}\,\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s+1)}{(s+\frac{1}{2})^2+1}\right]$$

$$= \frac{4}{5}\,\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{5}\,\mathcal{L}^{-1}\left[\frac{(s+\frac{1}{2})+\frac{1}{2}}{(s+\frac{1}{2})^2+1}\right]$$

$$= \frac{4}{5}\,\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{5}\,\mathcal{L}^{-1}\left[\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+1}\right] - \frac{4}{10}\,\mathcal{L}^{-1}\left[\frac{1}{(s+\frac{1}{2})^2+1}\right]$$

Recall: $\boxed{\mathcal{L}^{-1}\left[F(s-c)\right] = e^{ct} f(t)}$ $\qquad c = -\frac{1}{2}$

$$h(t) = \frac{4}{5}(1) - \frac{4}{5}\,e^{-t/2}\cos(t) - \frac{4}{10}\,e^{-t/2}\sin(t)$$

$$\boxed{h(t) = \frac{4}{5}\left[1 - e^{-t/2}\left(\cos(t) + \frac{1}{2}\sin(t)\right)\right]}$$

we conclude

$$\boxed{y(t) = h(t) - u(t-\pi)\,h(t-\pi)}$$

* **Example**: ((6.4) Probl. 10)

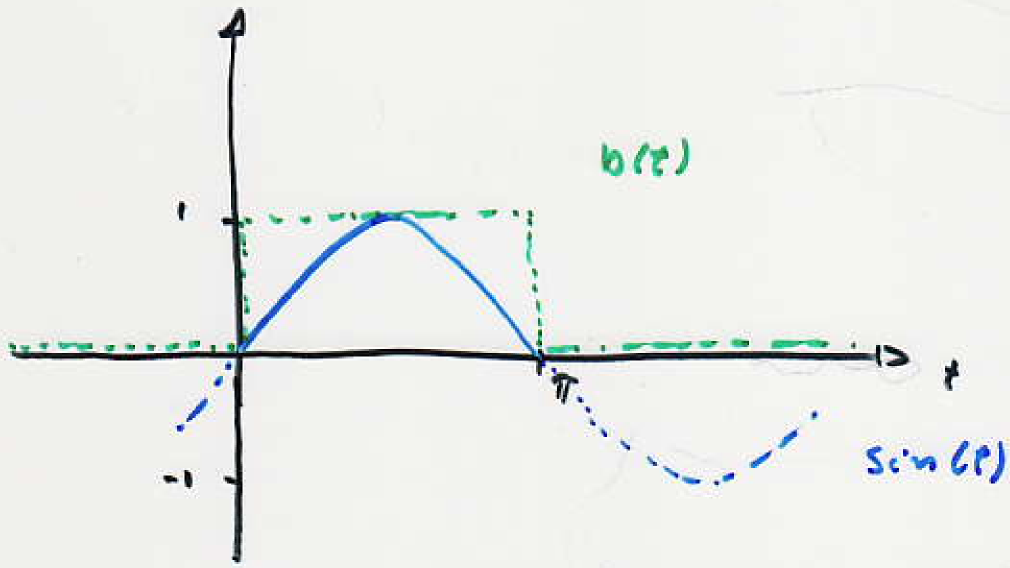
Find the sol. to IVP

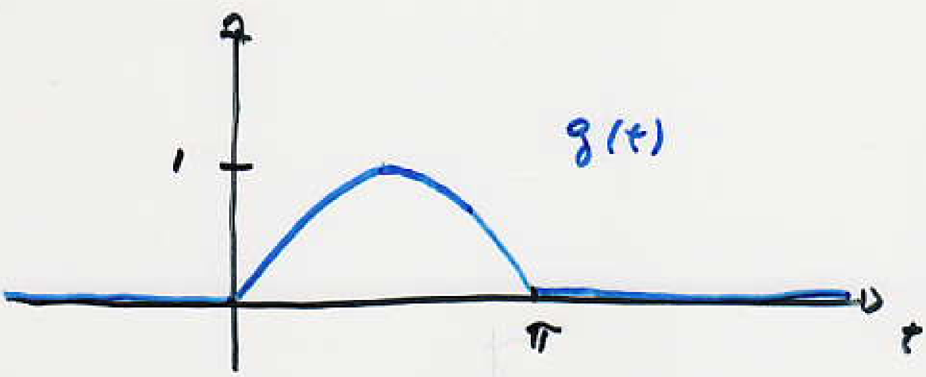
$$y'' + y' + \frac{5}{4} y = g(t)$$

$$y(0) = 0 \quad , \quad y'(0) = 0$$

$$g(t) = \begin{cases} \sin(t) & 0 \le t < \pi \\ 0 & t \geq \pi \end{cases}$$

Sol.





- Express $g(t)$ using step functions.

Recall: $b(t) = u(t) - u(t-\pi)$

So: $g(t) = [u(t) - u(t-\pi)] \sin(t)$

$$\boxed{g(t) = u(t)\sin(t) - u(t-\pi)\sin(t)}$$

Recall. $\sin(t-\pi) = -\sin(t)$

So:

$$\boxed{g(t) = u(t)\sin(t) + u(t-\pi)\sin(t-\pi)}$$

- Laplace transform the eq.

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\mathcal{L}[y] = \mathcal{L}[g]$$

- Initial conditions $y(0) = 0$, $y'(0) = 0$

$$s^2\mathcal{L}[y] + s\mathcal{L}[y] + \frac{5}{4}\mathcal{L}[y] = \mathcal{L}[g].$$

$$\left(s^2+s+\frac{5}{4}\right) \mathcal{L}[y] = \mathcal{L}[\,u(t)\sin(t)\,] + \mathcal{L}[\,u(t-\pi)\sin(t-\pi)\,]$$

$$\boxed{\left(s^2+s+\frac{5}{4}\right) \mathcal{L}[y] = \frac{1}{(s^2+1)} + \frac{e^{-\pi s}}{(s^2+1)}}$$

$$s^2+s+\frac{5}{4} = \left(s+\frac{1}{2}\right)^2+1$$ (complete the square.)

$$\boxed{\mathcal{L}[y] = \frac{1}{(s^2+1)\left[\left(s+\frac{1}{2}\right)^2+1\right]} + \frac{e^{-\pi s}}{(s^2+1)\left[\left(s+\frac{1}{2}\right)^2+1\right]}}$$

Denote: $$H(s) = \frac{1}{(s^2+1)\left[\left(s+\frac{1}{2}\right)^2+1\right]}$$

then

$$\boxed{\mathcal{L}[y] = H(s) + e^{-\pi s} H(s)}$$

If we find $\boxed{h(t) = \mathcal{L}^{-1}[H(s)]}$,

then the formula

$$y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s} H(s)]$$

implies that

$$\boxed{y(t) = h(t) + u(t-\pi)\, h(t)}$$

- <u>Find</u> <u>$h(t)$</u> <u>solution</u> <u>of</u>

$$\boxed{h(t) = \mathcal{L}^{-1}\left[\frac{1}{(s^2+1)\left[(s+\frac{1}{2})^2+1\right]} \right]}$$

- Partial fractions

$$\boxed{H(s) = \frac{1}{(s^2+1)\left[(s+\frac{1}{2})^2+1\right]} = \frac{as+b}{(s+\frac{1}{2})^2+1} + \frac{cs+d}{s^2+1}}$$

Find a, b, c, d

The result: $\boxed{a = \frac{16}{17}, \quad b = \frac{12}{17}, \quad c = -\frac{16}{17}, \quad d = \frac{4}{17}}$

$$H(s) = \frac{4}{17}\left[\frac{4s+3}{(s+\frac{1}{2})^2+1} + \frac{(-4s+1)}{s^2+1}\right]$$

$$H(s) = \frac{4}{17}\left[\frac{4(s+\frac{1}{2}-\frac{1}{2})+3}{(s+\frac{1}{2})^2+1} - 4\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

$$\boxed{H(s) = \frac{4}{17}\left[4\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+1} + \frac{1}{(s+\frac{1}{2})^2+1} - 4\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]}$$

<u>Recall:</u> $$F(s-c) = \mathcal{L}\left[e^{ct} f(t) \right]$$

$$H(s) = \frac{4}{17}\Big[4\,\mathcal{L}\left[e^{-t/2} \cos(t) \right]$$
$$+ \mathcal{L}\left[e^{-t/2} \sin(t) \right]$$
$$-4\,\mathcal{L}\left[\cos(t) \right] + \mathcal{L}\left[\sin(t) \right] \Big]$$

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big[4\,e^{-t/2} \cos(t) + e^{-t/2} \sin(t)$$
$$- 4\cos(t) + \sin(t) \Big] \Big]$$

$$H(s) = \mathcal{L}\left[h(t) \right]$$

$$\boxed{h(t) = \frac{4}{17}\left[e^{-t/2}\left(4\cos(t) + \sin(t)\right) - 4\cos(t) + \sin(t) \right]}$$

so the sol. of the diff. eq. is

$$\boxed{y(t) = h(t) + u(t-\pi)\, h(t-\pi)}$$