mth 235 L22

Plan:

* Step functions

* Piecewise discontinuous functions

* $\mathcal{L}$[ discont. funct. ]

* Properties of $\mathcal{L}$[ ].

(6.3)

* Overview

The Laplace transform method can be used to solve differential eqs. with discontinuous sources.

* Notation

If $\mathcal{L}[f(t)] = F(s)$,

then we denote

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

One can show that for a particular type of functions $f$, the notation above is well-defined.

* Example: We know:

$$\boxed{\mathcal{L}[e^{at}] = \frac{1}{s-a}} \qquad s > a,$$

Then:

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$$

(Recall table Page 317, textbook.)

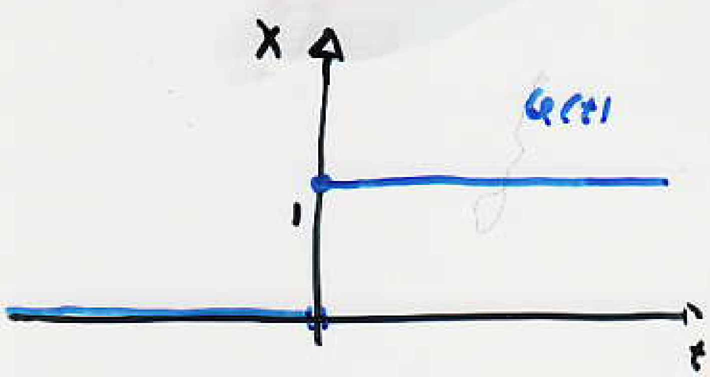
## * The step function

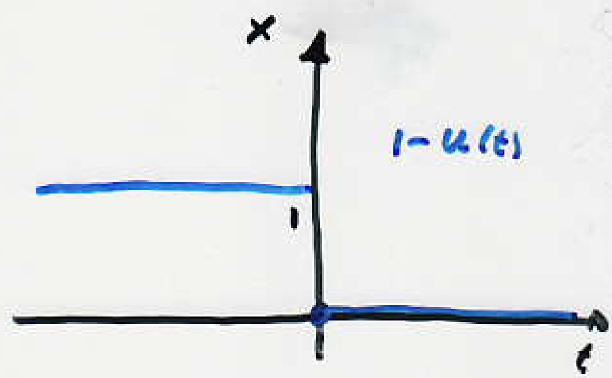
**Def:** The function

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

is called a **step function**.

## * Examples




## * Recall:

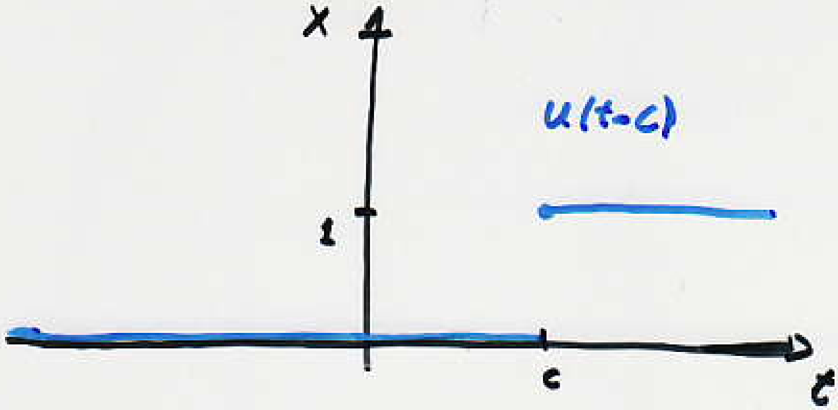
Given any positive number $c \in \mathbb{R}$, the function:

$u(t-c)$ right translation

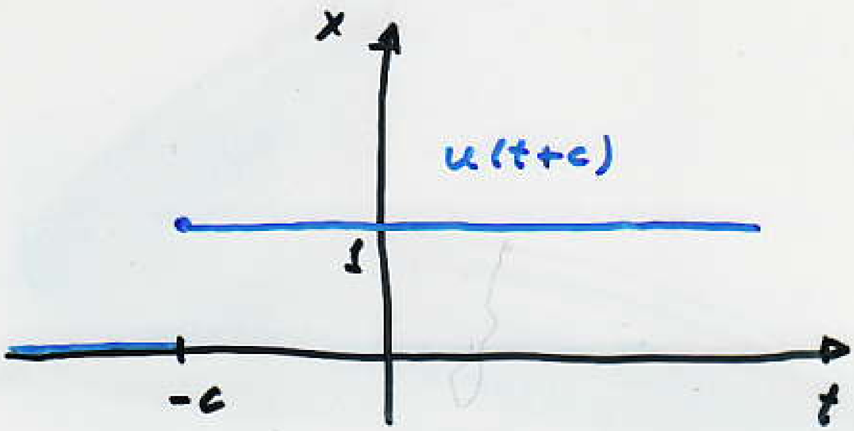
$u(t+c)$ left translation

of function $u(t)$.

* Examples



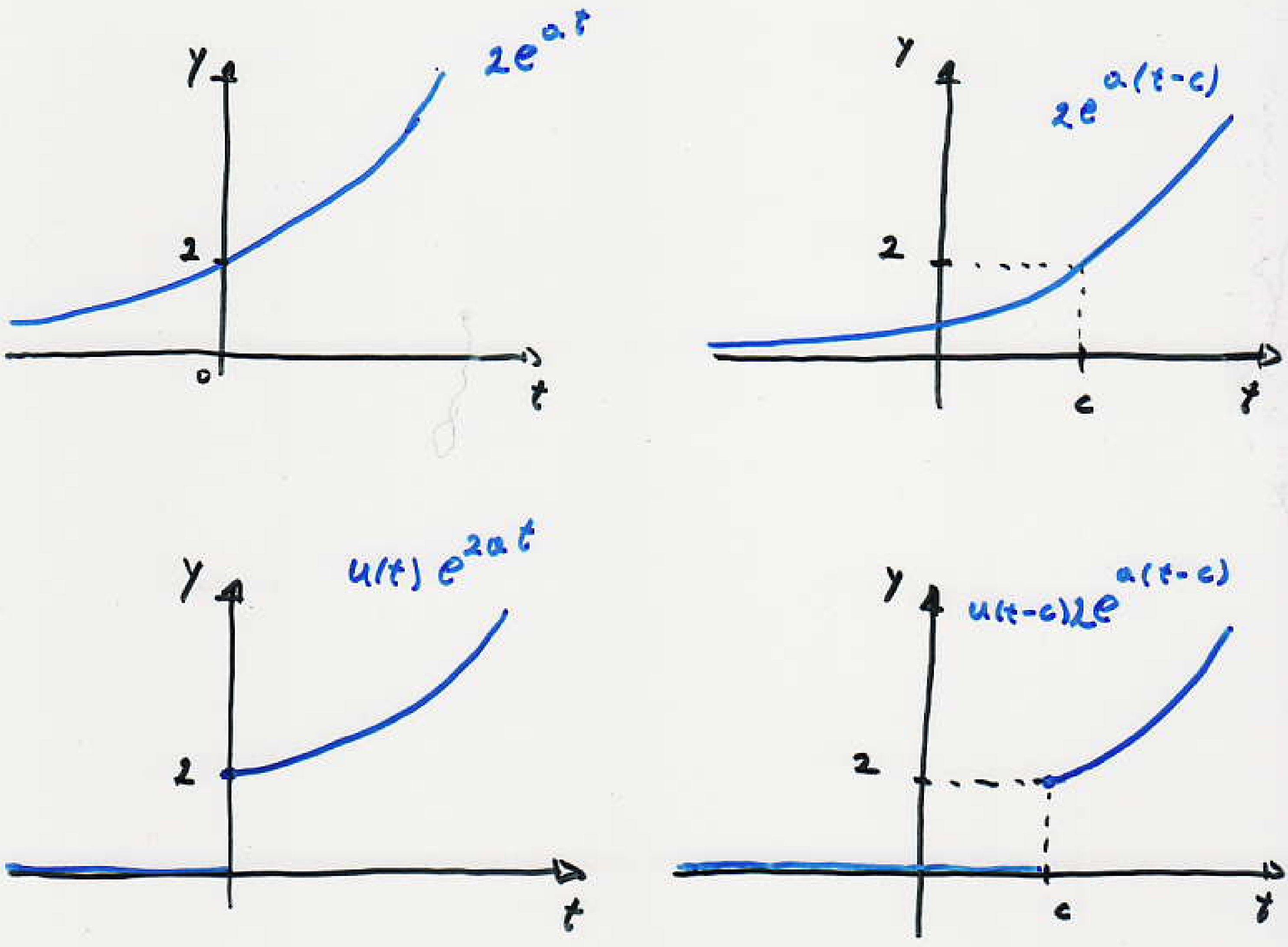

$$u(t-c) = \begin{cases} 0 & t-c<0 \\ 1 & t-c \geq 0 \end{cases}$$




$$u(t+c) = \begin{cases} 0 & t+c<0 \\ 1 & t+c \geq 0 \end{cases}$$

* **Remark**: Given $f(t)$ and $c>0$, then
  - $f(t-c)$ right translation
  - $f(t+c)$ left translation.

* **Examples**

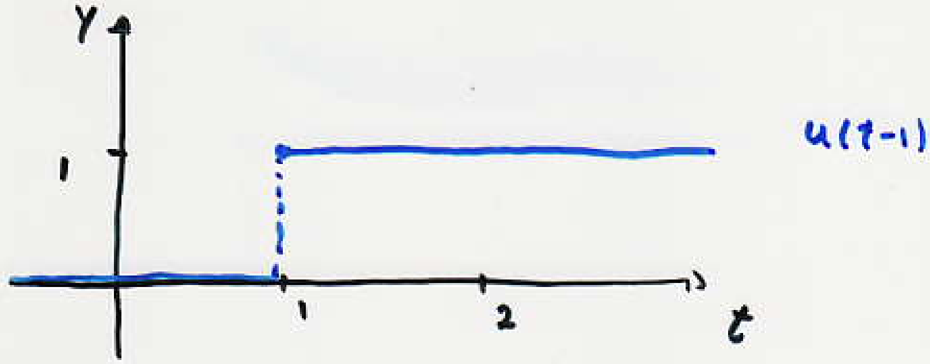


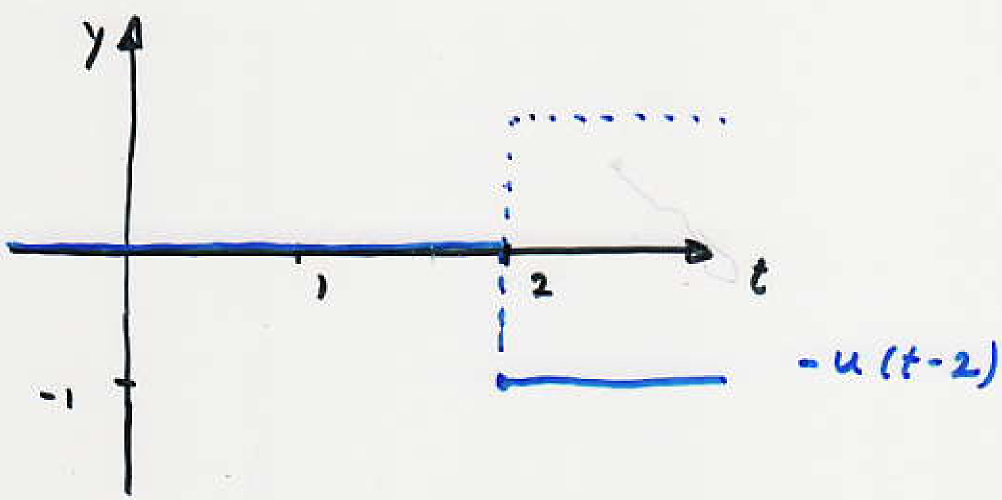
* **Examples** : [ Find the graph of

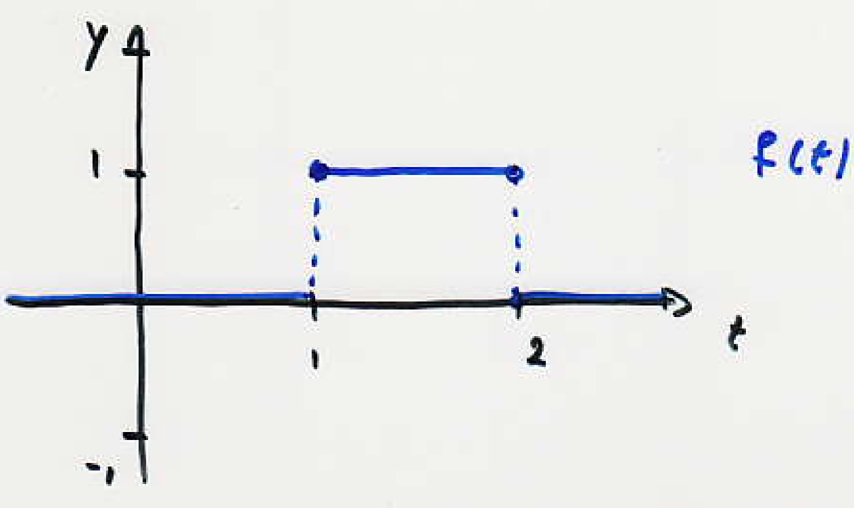
$$f(t) = u(t-1) - u(t-2)$$ ]

(unit bump)

Sol:



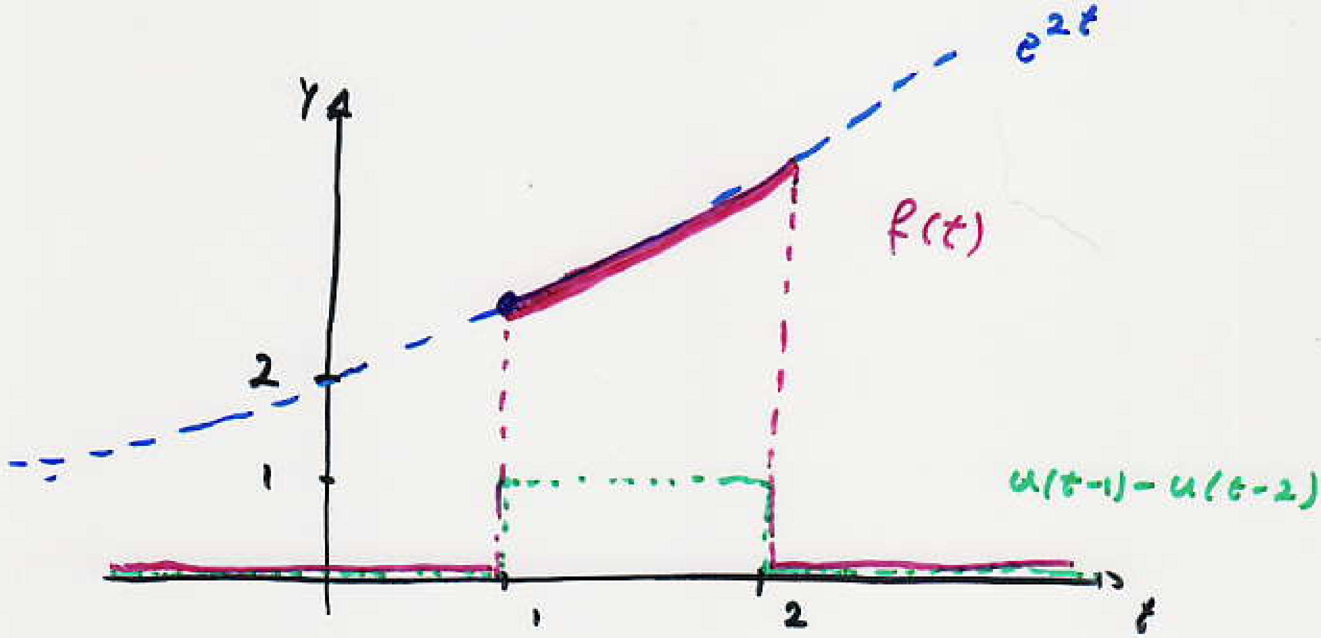



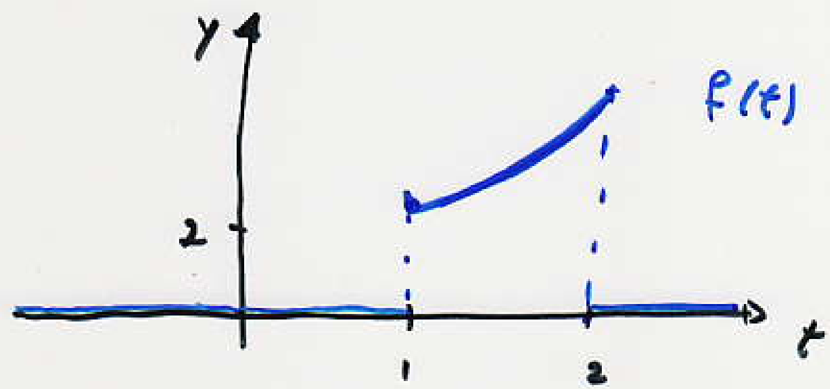



* **Example** [ Find the graph of

$$f(t) = e^{2t}\,[u(t-1) - u(t-2)]$$ ]

Sol:




* **Notation** : In the text book

$\boxed{u(t-c)}$ is denoted as

$\boxed{u_c(t)}$

* Laplace transform of step functions

Thm: $$\boxed{\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \quad s>0.}$$

Proof:

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} \, u(t-c) \, dt$$

$$= \int_c^\infty e^{-st} \, dt$$

$$= \lim_{N\to\infty} -\frac{1}{s} e^{-st}\Big|_c^N$$

$$= -\frac{1}{s} \lim_{N\to\infty} \left( e^{-sN} - e^{-cs} \right)$$

$$= \frac{e^{-cs}}{s} \qquad s>0.$$

∎

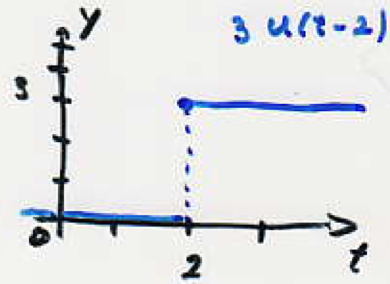
**Example:** compute $\mathcal{L}[3\,u(t-2)]$.

**Sol:**

$$\mathcal{L}[3\,u(t-2)] = 3\,\mathcal{L}[u(t-2)]$$

$$= 3\,\frac{e^{-2s}}{s}$$

$$\boxed{\mathcal{L}[3\,u(t-2)] = 3\,\frac{e^{-2s}}{s}}$$

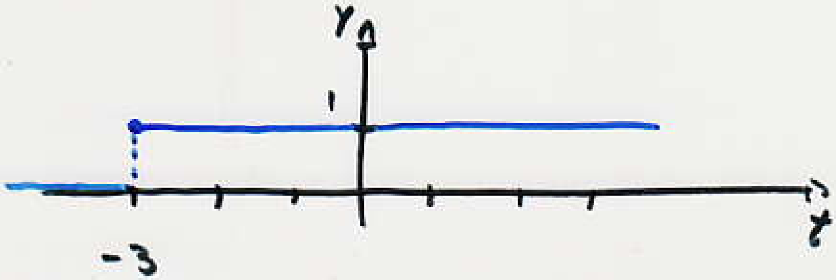



**Example:** compute $\mathcal{L}^{-1}\left[\frac{e^{3s}}{s}\right]$.

**Sol:**

$$\mathcal{L}^{-1}\left[\frac{e^{3s}}{s}\right] = \mathcal{L}^{-1}\left[\frac{e^{-(-3)s}}{s}\right]$$

$$\boxed{\mathcal{L}\left[\frac{e^{3s}}{s}\right] = u(t+3)}$$



* **Properties of the $\mathcal{L}[\ ]$.**

Thrm: If $\mathcal{L}[f(t)] = F(s)$ exists for $s > a \geq 0$, and if $c > 0$, then

$$\boxed{\mathcal{L}[u(t-c)\, f(t-c)] = e^{-cs} F(s),} \quad s > a,$$

Furthermore

$$\boxed{\mathcal{L}[e^{ct} f(t)] = F(s-c)} \quad s > a+c.$$

Comment:

| |
|---|
| $\mathcal{L}[\text{translation}(uf)] = \exp \times \mathcal{L}[f]$ |
| $\mathcal{L}[\exp \times f] = \text{translation}(\mathcal{L}[f])$ |

Equivalent notation:

| |
|---|
| $\mathcal{L}[u(t-c)\, f(t-c)] = e^{-cs}\, \mathcal{L}[f(t)](s)$ |
| $\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)](s-c)$ |

* <u>Example</u> [ compute $\mathcal{L}[u(t-2)\sin(a(t-2))]$. ]

<u>Sol.</u>

Recall : $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$

therefore :
$$\boxed{\mathcal{L}[u(t-2)\sin(a(t-2))] = e^{-2s}\dfrac{a}{s^2+a^2}}$$

* <u>Example</u> [ compute $\mathcal{L}[e^{3t}\sin(at)]$ ]

<u>Sol.</u>
$$\boxed{\mathcal{L}[e^{3t}\sin(at)] = \dfrac{a}{(s-3)^2+a^2}} \quad s>3.$$

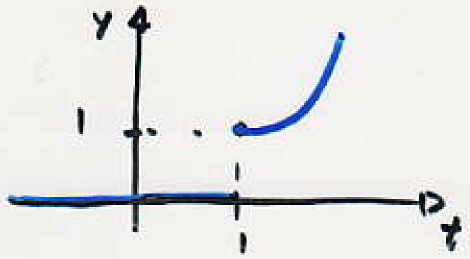
* <u>Example</u> Find the Laplace Transform of

$$f(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases}$$

<u>Sol.</u>

<u>Notice</u> :

$$t^2 - 2t + 2 = (t^2 - 2t + 1) + 1 = (t-1)^2 + 1$$




$$f(t) = u(t-1)\left[(t-1)^2 + 1\right]$$

$$\boxed{f(t) = u(t-1)(t-1)^2 + u(t-1)}$$

<u>Recall</u> :

$$\mathcal{L}[u(t-1)] = \frac{e^{-s}}{s} \quad ; \quad \mathcal{L}[t^2] = \frac{2!}{s^3}$$

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)(t-1)^2] + \mathcal{L}[u(t-1)]$$

$$= e^{-s}\frac{2}{s^3} + \frac{e^{-s}}{s}$$

$$\boxed{\mathcal{L}[f(t)] = \frac{e^{-s}}{s^3}(s^2+2)}$$

* **Remark**: The inverse of the formulas in the Thrm are

$$\boxed{\mathcal{L}^{-1}\left[e^{-cs}F(s)\right] = u(t-c)\, f(t-c)}$$

$$\boxed{\mathcal{L}^{-1}\left[F(s-c)\right] = e^{ct} f(t), \quad s>c.}$$

* **Example**: Find $\mathcal{L}^{-1}\left[\dfrac{e^{-4s}}{s^2+9}\right]$

**Sol**:

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+9}\right] = \frac{1}{3}\,\mathcal{L}^{-1}\left[e^{-4s}\left(\frac{3}{s^2+3^2}\right)\right]$$

**Recall**: $\mathcal{L}^{-1}\left[\dfrac{3}{s^2+3^2}\right] = \sin(3t)$

so:

$$\boxed{\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+9}\right] = \frac{1}{3}\, u(t-4)\, \sin\left[3(t-4)\right]}$$

* **Example** : [ Find $\mathcal{L}^{-1}\left[\dfrac{s-2}{(s-2)^2+9}\right]$. ]

**Sol:**

**Recall** : $$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at).$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{(s-2)}{(s-2)^2+3^2}\right] = e^{2t}\cos(3t)}$$

* **Example** : [ Find $\mathcal{L}^{-1}\left[\dfrac{2e^{-3s}}{s^2-4}\right]$ ]

**Sol:**

**Recall** : $$\mathcal{L}\left[\operatorname{Sh}(at)\right] = \frac{a}{s^2-a^2}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{2e^{-3s}}{s^2-4}\right]} = \mathcal{L}^{-1}\left[e^{-3s}\left(\frac{2}{s^2-2^2}\right)\right]$$

$$\boxed{= u(t-3)\ \operatorname{Sh}(2(t-3))}$$

* **Example** : $\left[ \text{Find} \quad \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2+s-2} \right] \right]$

**Sol.**

roots of $\quad P(s) = s^2 + s - 2$

$$s_{\pm} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \Rightarrow \boxed{s_+ = 1} \; \boxed{s_- = -2}$$

$$\boxed{s^2+s-2 = (s-1)(s+2)}$$

$$\boxed{\mathcal{L}^{-1}\left[ \frac{e^{-2s}}{s^2+s-2} \right] = \mathcal{L}^{-1}\left[ \frac{e^{-2s}}{(s-1)(s-2)} \right]}$$

Partial fraction method.

$$\frac{1}{(s-1)(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)}$$

$$= \frac{a(s+2) + b(s-1)}{(s-1)(s+2)}$$

$$\boxed{1 = (a+b)s + (2a-b)}$$

$$\boxed{\begin{array}{l} a+b=0 \\ 2a-b=1 \end{array}} \qquad \begin{array}{l} b=-a \\ 2a+a=1 \end{array} \quad \Rightarrow \quad \boxed{a=\tfrac{1}{3}} \quad \boxed{b=-\tfrac{1}{3}}$$

$$\boxed{\frac{1}{(s-1)(s+2)} = \frac{1}{3}\frac{1}{(s-1)} - \frac{1}{3}\frac{1}{(s+2)}}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+s-2}\right] = \mathcal{L}^{-1}\left[\frac{1}{3}\frac{e^{-2s}}{(s-1)} - \frac{1}{3}\frac{e^{-2s}}{(s+2)}\right]$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s-1)}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s+2)}\right]$$

<u>Recall</u>: $\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+s-2}\right] = \frac{1}{3}\, u(t-2)\, e^{(t-2)} - \frac{1}{3}\, u(t-2)\, e^{-2(t-2)}$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+s-2}\right] = \frac{1}{3}\, u(t-2)\left[e^{(t-2)} - e^{-2(t-2)}\right]}$$