mth 235 L21

Plan:
* Review Exam 2.
* Sections 3.1 - 3.6
  5.2, 5.4, 5.5.
* 6 problems, 50 minutes
* Practice exams.

* Exam : October 7 , 2008

(1) $$\left[\begin{array}{ll} \text{IVP:} & 2y'' + 8y' + 10y = 0 \\ & y(0) = 1 , \quad y'(0) = 0 \end{array}\right]$$

Sol:

$$2r^2 + 8r + 10 = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2}$$

$$\boxed{r_{\pm} = -2 \pm i}$$

$$y(t) = e^{-2t} \left(c_1 \cos(t) + c_2 \sin(t) \right)$$

$$y'(t) = -2 e^{-2t} \left(c_1 \cos(t) + c_2 \sin(t) \right)$$

$$+ e^{-2t} \left(-c_1 \sin(t) + c_2 \cos(t) \right)$$

Initial conditions:

$$1 = y(0) = c_1 \quad \Rightarrow \quad \boxed{c_1 = 1}$$

$$0 = y'(0) = -2c_1 + c_2 \quad \Rightarrow \quad \boxed{c_2 = 2}$$

$$\boxed{y(t) = e^{-2t}\left(\cos(t) + 2\sin(t)\right)}$$

(2) $\left[\begin{array}{l} \text{general sol. of} \\ y'' + 4y = 3\sin(2x) \end{array} \right]$

Sol.

undetermined coefficients.

homog. eq. $r^2 + 4 = 0 \Rightarrow \boxed{r_\pm = \pm 2i}$

$$\boxed{\begin{array}{l} y_1(x) = \cos(2x) \\ y_2(x) = \sin(2x) \end{array}}$$

$\tilde{y}_p = k_1 \sin(2x) + k_2 \cos(2x)$ wrong guess

$$\boxed{y_p = x\left(k_1 \sin(2x) + k_2 \cos(2x)\right)}$$

$$\boxed{\begin{array}{l} y_p' = \left[k_1 \sin(2x) + k_2 \cos(2x)\right] \\ \qquad + 2x\left[k_1 \cos(2x) - k_2 \sin(2x)\right] \end{array}}$$

$$\boxed{\begin{aligned} y_p'' &= 4\left[k_1 \cos(2x) - k_2 \sin(2x) \right] \\ &\quad - 4x \left[k_1 \sin(2x) + k_2 \cos(2x) \right] \end{aligned}}$$

$$y_p'' + 4 y_p = 3 \sin(2x)$$

$$\begin{aligned} &4 \left[k_1 \cos(2x) - k_2 \sin(2x) \right] \\ &- 4x \left[k_1 \sin(2x) + k_2 \cos(2x) \right] \\ &+ 4x \left[k_1 \sin(2x) + k_2 \cos(2x) \right] = 3 \sin(2x) \end{aligned}$$

$$\boxed{4 \left[k_1 \cos(2x) - k_2 \sin(2x) \right] = 3 \sin(2x)}$$

$x = 0 \qquad 4k_1 = 0 \qquad \Rightarrow \boxed{k_1 = 0}$

$2x = \frac{\pi}{2} \qquad -4k_2 = 3 \qquad \Rightarrow \boxed{k_2 = -\frac{3}{4}}$

$$\boxed{y_p = -\frac{3}{4} x \sin(2x)}$$

$$\boxed{y(x) = c_1 \cos(2x) + \left(c_2 - \frac{3}{4} x\right) \sin(2x)}$$

(3) Particular sol. of

$$(1-x)\, y'' + x\, y' - y = 2\,(1-x)^2 e^x, \quad x \neq 1$$

$y_1 = e^x \quad , \quad y_2 = x \qquad$ sol. hom. eq.

Sol.:

$$y'' + \frac{x}{(1-x)}\, y' - \frac{1}{(1-x)}\, y = 2\,(1-x)\, e^x$$

$$\boxed{g(x) = 2\,(1-x)\, e^x}$$

$$W = \begin{vmatrix} e^x & e^x \\ x & 1 \end{vmatrix} = e^x - x e^x$$

$$\boxed{W = (1-x)\, e^x}$$

$$u_1' = -\frac{y_2\, g}{W}$$

$$= -\frac{x\; 2(1-x)e^x}{(1-x)\; e^x} = -2x$$

$$u_1' = -2x \quad \Rightarrow \quad \boxed{u_1 = -x^2}$$

$$u_2' = \frac{y_1 \; g}{W}$$

$$= \frac{e^x \; 2(1-x)e^x}{(1-x)e^x} = 2e^x$$

$$u_2' = 2e^x \quad \Rightarrow \quad \boxed{u_2 = 2e^x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = -x^2 e^x + 2e^x \, x$$

$$\boxed{y_p = (-x^2 + 2x)\, e^x}$$

(4) $\left[\begin{array}{l} x_0 = 3 \ , \ \text{recurrence relation} \\ 2y'' + (x+1)\,y' + 3y = 0 \end{array} \right]$

Sol:

$x_0 = 3$: regular point.

$$\left[\; y(x) = \sum_{n=0}^{\infty} a_n (x-3)^n \; \right]$$

$$\left[\; y'(x) = \sum_{n=0}^{\infty} n\, a_n (x-3)^{n-1} \; \right]$$

$$\left[\; y''(x) = \sum_{n=0}^{\infty} n(n-1)\, a_n (x-3)^{n-2} \; \right]$$

$$(x+1)\,y' = (x+1) \sum_{n=0}^{\infty} n\, a_n (x-3)^{n-1}$$

$$= (x-3+3+1) \sum_{n=0}^{\infty} n\, a_n (x-3)^{n-1}$$

$$= (x-3) \sum_{n=0}^{\infty} n\, a_n (x-3)^{n-1}$$

$$+ 4 \sum_{n=0}^{\infty} n\, a_n (x-3)^{n-1}$$

$$(x+1)\,y' = \sum_{n=0}^{\infty} n\,a_n (x-3)^n + \sum_{n=1}^{\infty} 4\,n\,a_n (x-3)^{n-1}$$

$n-1=m$

$$\boxed{(x+1)\,y' = \sum_{n=0}^{\infty} n\,a_n (x-3)^n + \sum_{n=0}^{\infty} 4\,(n+1)\,a_{n+1} (x-3)^n}$$

$$\boxed{3y = \sum_{n=0}^{\infty} 3\,a_n (x-3)^n}$$

$$2y'' = \sum_{n=0}^{\infty} 2\,n(n-1)\,a_n (x-3)^{n-2}$$

$$= \sum_{n=2}^{\infty} 2\,n(n-1)\,a_n (x-3)^{n-2}$$

$n-2=m$

$$\boxed{2y'' = \sum_{n=0}^{\infty} 2\,(n+2)(n+1)\,a_{n+2} (x-3)^n}$$

$$\sum_{n=0}^{\infty} \Big[2\,(n+2)(n+1)\,a_{n+2} + n\,a_n + 4\,(n+1)\,a_{n+1}$$

$$+ 3\,a_n \Big] (x-3)^n = 0$$

$$\boxed{2(n+2)(n+1)\,a_{n+2} + 4(n+1)\,a_{n+1} + (n+3)\,a_n = 0}$$

recurrence relation. $n \geq 0$.

Find the first 4 terms:

$$a_{n+2} = \frac{-4(n+1)\,a_{n+1} - (n+3)\,a_n}{2(n+2)(n+1)}.$$

$n=0$ $$a_2 = \frac{-4a_1 - 3a_0}{4} \Rightarrow \boxed{a_2 = -a_1 - \frac{3}{4}a_0}$$

$n=1$ $$a_3 = \frac{-4(2)\,a_2 - 4a_1}{2(3)(2)}$$

$$a_3 = -\frac{2}{3}a_2 - \frac{4}{3}a_1$$

$$a_3 = +\frac{2}{3}a_1 + \frac{2}{3}\frac{3}{4}a_0 - \frac{4}{3}a_1$$

$$\boxed{a_3 = -\frac{2}{3}a_1 + \frac{1}{2}a_0}$$

$$y(x) = a_0 + a_1 (x-3) + a_2 (x-3)^2 + a_3 (x-3)^3 + \ldots$$

$$= a_0 + a_1 (x-3) + \left(-a_1 - \frac{3}{4} a_0\right)(x-3)^2$$

$$+ \left(-\frac{2}{3} a_1 + \frac{1}{2} a_0\right)(x-3)^3 + \ldots$$

$$\boxed{\begin{aligned} y(x) &= a_0 \left[1 - \frac{3}{4}(x-3)^2 + \frac{1}{2}(x-3)^3 + \ldots \right] \\ &\quad + a_1 \left[(x-3) - (x-3)^2 - \frac{2}{3}(x-3)^3 + \ldots \right] \end{aligned}}$$

(5) [general sol of.

$$x^2 y'' + 3x y' + 5y = 0$$]

Sol.

Euler eq. $\Rightarrow$ $y(x) = |x|^r$.

$$r(r-1) + 3r + 5 = 0$$

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$\boxed{r_\pm = -1 \pm 2i}$$

$$\boxed{y(x) = \frac{1}{x}\left[c_1 \cos(2 \ln|x|) + c_2 \sin(2 \ln|x|)\right]}$$

(1) Exam 3 Nov. 11, 2008

Indicial eq, recurrence rel.
First 3 terms, $x_0 = 0$ r.s.p.

$$x^2 y'' + \left(x^2 + \frac{1}{4}\right) y = 0$$

Sol:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r-1)}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$$

$$x^2 y''(x) = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r-2)}$$

$$\boxed{x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}}$$

$$\left(x^2+\tfrac{1}{4}\right) y = \left(x^2+\tfrac{1}{4}\right) \sum_{n=0}^{\infty} a_n x^{(n+r)}$$

$$= x^2 \sum_{n=0}^{\infty} a_n x^{(n+r)} + \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{(n+r)}$$

$$= \sum_{n=0}^{\infty} a_n x^{(n+r+2)} + \sum_{n=0}^{\infty} \frac{a_n}{4} x^{n+r}$$

$m = n+2$

$$\left(x^2+\tfrac{1}{4}\right) y = \sum_{n=2}^{\infty} a_{n-2} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{a_n}{4} x^{(n+r)}$$

$$\left[\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{(n+r)} + \sum_{n=2}^{\infty} a_{n-2} x^{(n+r)} + \sum_{n=0}^{\infty} \frac{a_n}{4} x^{(n+r)} = 0 \right]$$

$$\left[r(r-1)\, a_0 x^r + (r+1)r\, a_1 x^{(r+1)} + \frac{a_0}{4} x^r + \frac{a_1}{4} x^{(r+1)} + \sum_{n=2}^{\infty} \left[(n+r)(n+r-1)\, a_n + \frac{a_n}{4} + a_{n-2} \right] x^{(n+r)} = 0 \right]$$

$$\boxed{r(r-1) + \frac{1}{4} = 0}$$

$$\boxed{(r+1)r + \frac{1}{4} = 0}$$

$$\boxed{\left[(n+r)(n+r-1) + \frac{1}{4} \right] a_n + a_{n-2} = 0}$$

$$r^2 - r + \frac{1}{4} = 0 \qquad r_1 = \frac{1 \pm \sqrt{1-1}}{2} \Rightarrow \boxed{r_1 = \frac{1}{2}}$$

$$r^2 + r + \frac{1}{4} = 0 \qquad r_2 = \frac{-1 \pm \sqrt{1-1}}{2} \Rightarrow \boxed{r_2 = -\frac{1}{2}}$$

Choose $\boxed{r_1 = \frac{1}{2}}$

$$\left[\left(n+\tfrac{1}{2}\right)\left(n+\tfrac{1}{2}-1\right) + \tfrac{1}{4}\right] a_n + a_{n-2} = 0$$

$$\left[\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right) + \tfrac{1}{4}\right] a_n + a_{n-2} = 0$$

$$\left[n^2 - \tfrac{1}{4} + \tfrac{1}{4}\right] a_n + a_{n-2} = 0$$

$$\boxed{n^2 a_n + a_{n-2} = 0}$$

recurrence rel.

$n \geq 2$. n even

$$a_n = -\frac{a_{n-2}}{n^2}$$

$n=2$ $\boxed{a_2 = -\frac{a_0}{4}}$

$n=4$ $a_4 = -\frac{a_2}{4^2} \Rightarrow \boxed{a_4 = \frac{a_0}{4^2(4)}}$

$$y = x^{r_1}\left(a_0 + a_2 x^2 + a_4 x^4 + \dots\right)$$

$$\boxed{y = a_0 x^{1/2}\left[1 - \frac{x^2}{4} + \frac{x^4}{4^3} - \dots\right]}$$

* Exam 2 Oct. 8, 2008.

(3) [Variation of parameters.

$y'' + 4y' + 4y = x^{-2} e^{-2x}$]

Sol:

$$\boxed{g(x) = x^{-2} e^{-2x}}$$

Sol. of homog. eq.

$$r^2 + 4r + 4 = 0$$

$$r = \frac{-4 \pm \sqrt{16-16}}{2} = -2.$$

$$\boxed{r_\pm = -2}$$

$$\boxed{y_1(x) = e^{-2x}} \qquad \boxed{y_2(x) = x\, e^{-2x}}$$

$$W = \begin{vmatrix} e^{-2x} & -2e^{-2x} \\ x e^{-2x} & e^{-2x} - 2x e^{-2x} \end{vmatrix}$$

$$= e^{-4x}(1-2x) + 2x e^{-4x}$$

$$\boxed{W = e^{-4x}}$$

$$u_1' = -\frac{y_2 g}{W} = -\frac{x e^{-2x}\left(x^{-2} e^{-2x}\right)}{e^{-4x}}$$

$$u_1' = -\frac{1}{x} \Rightarrow \boxed{u_1 = -\ln|x|}$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{-2x}\left(x^{-2} e^{-2x}\right)}{e^{-4x}}$$

$$u_2' = x^{-2} \Rightarrow \boxed{u_2 = -\frac{1}{x}}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -\ln|x|\, e^{-2x} + \left(-\frac{1}{x}\right) x\, e^{-2x}$$

$$\boxed{y_p = -\ln|x|\, e^{-2x} - e^{-2x}}$$

$$\boxed{\tilde{y}_p = -\ln|x|\, e^{-2x}}$$